Light Propagation Through a Polarized Alkali Vapor in a Magnetic Field at an Arbitrary Angle

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Version 1.10

January 24, 2008

Abstract

We calculate the wavenumbers and polarization vectors for the two eigenmodes of light propagation through a polarized alkali vapor in a magnetic field. The wavenumbers correspond to the atomic polarizabilities for absorbing photons with ± 1 unit of angular momentum. If there is an angle θ between the light propagation direction and the magnetic field, then, to a very good approximation, the wavenumbers are modified by simply multiplying the alkali polarization $P_{\mathtt{A}}$ by $\cos(\theta)$. The polarization eigenvectors correspond to the circular polarization basis of light with a small admixture of linear polarization components both parallel and perpendicular to the light propagation direction.

Contents

1 Introduction

It is useful to know how light is effected by the presence of a polarized alkali vapor when (1) optical pumping or (2) measuring alkali polarization and densities using Faraday Rotation. We'll assume that the polarized alkali vapor is located within a magnetic field that orients the alkali angular momentum. In other words, the magnetic field provides one axis of a coordinate system that naturally describes the alkali atoms. The light and the alkali atoms are connected through the atomic polarizability, which itself is related to the dielectric permittivity. We'll start with Maxwell's equations and end with a matrix eigenvalue equation. The solution to this equation will give the wavenumbers and polarization vectors for the two eigenmodes of propagation through the vapor. An eigenmode of propagation has the property that its polarization vector does not change as it propagates through the alkali vapor.

In SI, Maxwell's equations are:

$$\begin{aligned}
\vec{\nabla} \cdot \vec{D} &= \rho & \vec{\nabla} \times \vec{E} &= -\partial \vec{B}/\partial t \\
\vec{\nabla} \cdot \vec{B} &= 0 & \vec{\nabla} \times \vec{H} &= \vec{J} + \partial \vec{D}/\partial t
\end{aligned} \tag{1}$$

and the constitutive relations for a linear medium between the electric vector $\vec{E}$ & the electric displacement $\vec{D}$ and between the magnetic induction $\vec{B}$ & the magnetic vector $\vec{H}$ are:

$$\vec{D} = \overleftrightarrow{\varepsilon} \cdot \vec{E} \qquad \vec{B} = \overleftrightarrow{\mu} \cdot \vec{H} \tag{2}$$

where $\overleftrightarrow{\varepsilon}$ & $\overleftrightarrow{\mu}$ are the dielectric permittivity & the magnetic permeability tensors, respectively. In our case, there are no free charges ρ and currents $\vec{J}$. Since the light has optical frequencies, the magnetic permeability tensor $\overleftrightarrow{\mu}$ is very nearly equal to the scalar free space value μ_0:

$$
\begin{aligned}
\vec{\nabla} \cdot \vec{D} = 0 && \vec{\nabla} \times \vec{E} = -\mu_0 \partial \vec{H}/\partial t \\
\vec{\nabla} \cdot \vec{H} = 0 && \vec{\nabla} \times \vec{H} = \partial \vec{D}/\partial t
\end{aligned}
\tag{3}
$$

Representing the spatial and time dependence of $\vec{E}$, $\vec{D}$, $\vec{B}$, & $\vec{H}$ in plane wave form, such as $\vec{E} \rightarrow \vec{E} \exp\left[i\vec{k} \cdot \vec{r} - i\omega t\right]$, results in:

$$
\begin{aligned}
\vec{k} \cdot \vec{D} = 0 && \vec{k} \times \vec{E} = \omega \mu_0 \vec{H} \\
\vec{k} \cdot \vec{H} = 0 && \vec{k} \times \vec{H} = -\omega \vec{D}
\end{aligned}
\tag{4}
$$

where $\vec{k}$ is the wavenumber and $\omega = 2\pi\nu$ is the frequency. Combining the two cross product equations results in:

$$
\vec{k} \times \left(\vec{k} \times \vec{E}\right) = \vec{k}\left(\vec{k} \cdot \vec{E}\right) - k^2 \vec{E} = -\omega^2 \mu_0 \vec{D}
\tag{5}
$$

where we've used the identity $\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \vec{b}\left(\vec{a} \cdot \vec{c}\right) - \vec{c}\left(\vec{a} \cdot \vec{b}\right)$. Finally, using the constitutive relation for $\vec{D}$, factoring out $\vec{E}$, and moving things around gives:

$$
\left(
\begin{bmatrix}
k^2 & 0 & 0 \\
0 & k^2 & 0 \\
0 & 0 & k^2
\end{bmatrix}
-
\begin{bmatrix}
k_1^2 & k_1 k_2 & k_1 k_3 \\
k_2 k_1 & k_2^2 & k_2 k_3 \\
k_3 k_1 & k_3 k_2 & k_3^2
\end{bmatrix}
- \omega^2 \mu_0 \overleftrightarrow{\varepsilon}
\right) \cdot \vec{E} = 0
\tag{6}
$$

The derivation of this equation can also be found in *Principles of Optics* by Born & Wolf (7th Edition, Chapter 15) and also *Optical Waves in Crystals: Propagation and Control of Laser Radiation* by Yariv & Yeh (Wiley Classics Library Edition, Chapter 4).

2   Eigenvalues

The following calculation is a detailed extension to the one found in "Skew light propagation in optically thick optical pumping cells", Chann et al, PRA 66 033406 (2002). There are in general three coordinate systems that one could use to solve for the possible eigenvalues for k:

1. the atomic basis, namely one that reflects the symmetry of the alkali vapor $\left(\hat{r}_+^*; \hat{r}_-^*; \hat{r}_0^* = \hat{B}_0\right)$ which is determined by the magnetic field $\vec{B}_0$

2. the linear $\left(\mathcal{P}; \mathcal{S}; \mathcal{Z} = \hat{k}\right)$ or circular $(\mathcal{R}; \mathcal{L}; \mathcal{Z})$ polarization basis of the light

3. the polarization eigenvector basis, namely the one for which the matrix multiplying $\vec{E}$ is diagonal

Unfortunately, we usually do not know beforehand what the polarization eigenvector basis is. However, when the magnetic field and light propagation direction point in the same direction, all three coordinates systems happen to coincide. Therefore we'll take advantage of this fact and choose to work in the circular polarization basis of the light. Consequently, the wave number dyad is represented as:

$$
\vec{k}\vec{k} =
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & k^2
\end{bmatrix}
\tag{7}
$$

where $k_1 = k_2 = 0$ and $\sqrt{k_1^2 + k_2^2 + k_3^2} = k^2$ due to the orthonormality of the circular polarization basis.

In the atomic basis $(\hat{r}_+^*; \hat{r}_-^*; \hat{r}_0^*)$, the dielectric tensor is, by construction, diagonal and given as:

$$\overleftrightarrow{\varepsilon} = \varepsilon_0 \left(1 + \frac{[A]}{\varepsilon_0} \begin{bmatrix} \alpha_+ & 0 & 0 \\ 0 & \alpha_- & 0 \\ 0 & 0 & \alpha_0 \end{bmatrix} \right) \tag{8}$$

where ε_0 is the dielectric permittivity of free space and $[A]$ is the alkali number density. The polarizabilities are given by (see my other technotes):

$$\alpha_q = \frac{\varepsilon_0 r_e c^2}{2\pi} \left[\underbrace{\frac{f_1/\nu_1}{\nu_1 - \nu - i\Gamma_1/2} (1 - qP_A)}_{\text{D1 transition}} + \underbrace{\frac{f_2/\nu_2}{\nu_2 - \nu - i\Gamma_2/2} \left(1 + \frac{qP_A}{2} \right)}_{\text{D2 transition}} \right] \tag{9}$$

where r_e is the classical electron radius, c is the speed of light in a vacuum, P_A is the alkali polarization, $q(=0, \pm1)$ is the amount of angular momentum transferred to an alkali atom, f_n is the oscillator strength, ν_n is the transition frequency, and Γ_n is the full width half maximum of the transition. The subscripts $n(=1,2)$ refer to the D1 and D2 transitions of the alkali atom.

We'll have to transform the polarizability tensor from the $(\hat{r}_+^*; \hat{r}_-^*; \hat{r}_0^*)$ basis to the $(\mathcal{R}; \mathcal{L}; \mathcal{Z})$ basis in the following way:

1. switch from the $(\hat{r}_+^*; \hat{r}_-^*; \hat{r}_0^*)$ basis to the $(\hat{x}; \hat{y}; \hat{z})$ basis relative to the atomic system

2. rotate by an angle θ from the $(\hat{x}; \hat{y}; \hat{z})$ basis relative to the atomic system to the $(\mathcal{P}; \mathcal{S}; \mathcal{Z})$ basis relative to the light polarization, where θ is the angle between the magnetic field and the light propagation direction as in $\hat{B}_0 \cdot \hat{k} = \cos(\theta)$

3. switch from the $(\mathcal{P}; \mathcal{S}; \mathcal{Z})$ basis to the $(\mathcal{R}; \mathcal{L}; \mathcal{Z})$ basis relative to the light polarization

We'll note the following relationships between the different bases relative to the light system:

$$\mathcal{R} = \frac{\mathcal{P} + i\mathcal{S}}{\sqrt{2}} \qquad\qquad \mathcal{L} = \frac{\mathcal{P} - i\mathcal{S}}{\sqrt{2}} \tag{10}$$

between the different bases relative to the atomic system:

$$\vec{r} = r_+ \hat{r}_+^* + r_- \hat{r}_-^* + r_0 \hat{r}_0^* = x\hat{x} + y\hat{y} + z\hat{z} \tag{11}$$

$$r_+ = -\left(\frac{x + iy}{\sqrt{2}} \right) \qquad r_- = \frac{x - iy}{\sqrt{2}} \qquad r_0 = z \tag{12}$$

and finally between the atomic and light systems:

$$\hat{x} = \mathcal{P} \cos(\theta) - \mathcal{Z} \sin(\theta) \qquad\qquad \hat{z} = \mathcal{P} \sin(\theta) + \mathcal{Z} \cos(\theta) \tag{13}$$

Switching the basis of the polarizability tensor from the circular basis to linear basis relative to the atomic system gives:

$$\overleftrightarrow{\alpha} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & +1 & 0 \\ -i & -i & 0 \\ 0 & 0 & +1 \end{bmatrix} \begin{bmatrix} \alpha_+ & 0 & 0 \\ 0 & \alpha_- & 0 \\ 0 & 0 & \alpha_0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & +i & 0 \\ +1 & +i & 0 \\ 0 & 0 & +1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \sigma & -i\delta & 0 \\ i\delta & \sigma & 0 \\ 0 & 0 & 2\alpha_0 \end{bmatrix} \tag{14}$$

where σ & δ are the sum & difference of α_+ & α_-:

$$\sigma = \alpha_+ + \alpha_- = 2\alpha_0 \qquad\qquad \delta = \alpha_+ - \alpha_- \tag{15}$$

To simplify things, we'll break the the resulting matrix up into the following:

$$\overleftrightarrow{\alpha} = \frac{1}{2} \begin{bmatrix} \sigma & -i\delta & 0 \\ i\delta & \sigma & 0 \\ 0 & 0 & 2\alpha_0 \end{bmatrix} = \frac{\sigma}{2} + i\frac{\delta}{2} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{16}$$

3

Now we rotate by θ from the linear basis of atomic system to linear polarization basis of the light:

$$
\begin{bmatrix}
\cos(\theta) & 0 & \sin(\theta) \\
0 & 1 & 0 \\
-\sin(\theta) & 0 & \cos(\theta)
\end{bmatrix}
\begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\cos(\theta) & 0 & -\sin(\theta) \\
0 & 1 & 0 \\
\sin(\theta) & 0 & \cos(\theta)
\end{bmatrix}
=
\begin{bmatrix}
0 & -\cos(\theta) & 0 \\
\cos(\theta) & 0 & -\sin(\theta) \\
0 & \sin(\theta) & 0
\end{bmatrix}
\tag{17}
$$

Finally we'll switch from the linear polarization basis of the light to the circular one:

$$
\frac{1}{\sqrt{2}}
\begin{bmatrix}
+1 & -i & 0 \\
+1 & +i & 0 \\
0 & 0 & +1
\end{bmatrix}
\begin{bmatrix}
0 & -\cos(\theta) & 0 \\
\cos(\theta) & 0 & -\sin(\theta) \\
0 & \sin(\theta) & 0
\end{bmatrix}
\frac{1}{\sqrt{2}}
\begin{bmatrix}
+1 & +1 & 0 \\
+i & -i & 0 \\
0 & 0 & +1
\end{bmatrix}
\tag{18}
$$

which gives:

$$
\begin{bmatrix}
-i\cos(\theta) & 0 & -\frac{i}{2}\sin(\theta) \\
0 & +i\cos(\theta) & +\frac{i}{2}\sin(\theta) \\
+\frac{i}{2}\sin(\theta) & -\frac{i}{2}\sin(\theta) & 0
\end{bmatrix}
\tag{19}
$$

Therefore, the polarizability in the circular polarization basis of the is:

$$
\overleftrightarrow{\alpha} = \frac{1}{2}
\begin{bmatrix}
\sigma + \delta\cos(\theta) & 0 & +\frac{\delta}{2}\sin(\theta) \\
0 & \sigma - \delta\cos(\theta) & -\frac{\delta}{2}\sin(\theta) \\
-\frac{\delta}{2}\sin(\theta) & +\frac{\delta}{2}\sin(\theta) & \sigma
\end{bmatrix}
=
\begin{bmatrix}
\alpha_{\mathcal{R}} & 0 & +\frac{\delta}{4}\sin(\theta) \\
0 & \alpha_{\mathcal{L}} & -\frac{\delta}{4}\sin(\theta) \\
-\frac{\delta}{4}\sin(\theta) & +\frac{\delta}{4}\delta\sin(\theta) & \alpha_0
\end{bmatrix}
\tag{20}
$$

and consequently the matrix equation is written as

$$
\begin{bmatrix}
k^2 - k_{\mathcal{R}}^2 & 0 & -\Delta_k^2 \\
0 & k^2 - k_{\mathcal{L}}^2 & +\Delta_k^2 \\
+\Delta_k^2 & -\Delta_k^2 & -k_0^2
\end{bmatrix} \cdot \vec{E} = 0
\tag{21}
$$

where we have defined the following quantities:

$$
k_{\mathcal{Q}}^2 = \frac{\omega^2}{c^2}\left(1 + \frac{[A]}{\varepsilon_0}\alpha_{\mathcal{Q}}\right) \qquad
\Delta_k^2 = \frac{\omega^2}{c^2}\frac{[A]}{\varepsilon_0}\left(\frac{\alpha_+ - \alpha_-}{4}\right)\sin(\theta)
\tag{22}
$$

$$
\alpha_{\mathcal{Q}} = \frac{\varepsilon_0 r_e c^2}{2\pi}\left[\frac{f_1/\nu_1}{\nu_1 - \nu - i\Gamma_1/2}\left(1 - qP_{\text{A}}\cos(\theta)\right) + \frac{f_2/\nu_2}{\nu_2 - \nu - i\Gamma_2/2}\left(1 + \frac{qP_{\text{A}}\cos(\theta)}{2}\right)\right]
\tag{23}
$$

where $q = +1, 0, -1$ for $\mathcal{Q} = \mathcal{R}, 0, \mathcal{L}$ respectively. This equation is solved by setting the determinant of the matrix to zero:

$$
\begin{vmatrix}
k^2 - k_{\mathcal{R}}^2 & 0 & -\Delta_k^2 \\
0 & k^2 - k_{\mathcal{L}}^2 & +\Delta_k^2 \\
+\Delta_k^2 & -\Delta_k^2 & -k_0^2
\end{vmatrix}
= \left(k^2 - k_{\mathcal{R}}^2\right)\left[-\left(k^2 - k_{\mathcal{L}}^2\right)k_0^2 + \Delta_k^4\right] + \Delta_k^4\left(k^2 - k_{\mathcal{L}}^2\right) = 0
\tag{24}
$$

This can be rearranged to:

$$
k^4 + k^2\left[-k_{\mathcal{L}}^2 - k_{\mathcal{R}}^2 - 2\Delta_k^4/k_0^2\right] + \left[k_{\mathcal{R}}^2 k_{\mathcal{L}}^2 + \left(k_{\mathcal{R}}^2 + k_{\mathcal{L}}^2\right)\Delta_k^4/k_0^2\right] = 0
\tag{25}
$$

and is then solved using the quadratic equation:

$$
k^2 = \frac{1}{2}\left[k_{\mathcal{R}}^2 + k_{\mathcal{L}}^2 + 2\Delta_k^4/k_0^2 \pm \sqrt{\left(k_{\mathcal{R}}^2 - k_{\mathcal{L}}^2\right)^2 + 4\Delta_k^8/k_0^4}\right]
\tag{26}
$$

The two eigenvalues for k^2 are given by the two solutions above. However, a more illuminating form for k^2 can be obtained if we compare the size of Δ_k^2 against two different scales:

$$
\frac{\Delta_k^2}{k_0^2} = \frac{[A]\left(\alpha_+ - \alpha_-\right)\sin(\theta)}{4\varepsilon_0\left(1 + \frac{[A]}{\varepsilon_0}\alpha_0\right)} \approx 10^{-3}\left(\frac{\sin(\theta)}{4}\right) \qquad\qquad
\frac{\Delta_k^2}{k_{\mathcal{R}}^2 - k_{\mathcal{L}}^2} = \frac{\tan(\theta)}{4}
\tag{27}
$$

4

where we have used the fact that the maximum absolute value of the polarizability occurs at resonance for the D2 transition $\nu = \nu_2$ which corresponds to:

$$\left[\frac{[A]}{\varepsilon_0}\alpha_0\right]_{\text{max}} = \left[\frac{[A]}{\varepsilon_0}\left(\frac{\alpha_+ - \alpha_-}{P_{\text{A}}}\right)\right]_{\text{max}} = \frac{[A]r_e c\lambda_2 f_2}{\pi\Gamma_2} = 10^{-3} \times \left(\frac{[A]}{10^{15}\ \text{cm}^{-3}}\right)\left(\frac{\lambda_2}{780\ \text{nm}}\right)\left(\frac{140\ \text{GHz}}{\Gamma_2}\right) \tag{28}$$

and we have put in typical values for the alkali density and absorbtion linewidth. Therefore, the angle for which the two terms under the square root are equal, for $P_{\text{A}} = 1$, is about $\theta = 89.993$. In almost all cases, the first term dominates over the second term under the square root and we can write the two solutions as:

$$k^2 = \left(k_{\mathcal{R}}^2\ \text{or}\ k_{\mathcal{L}}^2\right) + \Delta_k^4/k_0^2 \tag{29}$$

3 Eigenvectors

The polarization eigenvectors that correspond to these wavenumber eigenvalues are found by solving the following system of equations:

$$\begin{aligned}
\left(k^2 - k_{\mathcal{R}}^2\right)E_1 - \Delta_k^2 E_3 &= 0 \\
\left(k^2 - k_{\mathcal{L}}^2\right)E_2 + \Delta_k^2 E_3 &= 0 \\
\Delta_k^2 E_1 - \Delta_k^2 E_2 - k_0^2 E_3 &= 0
\end{aligned} \tag{30}$$

which can be rearranged to give the following useful ratios among the components of the eigenvectors:

$$\left(k^2 - k_{\mathcal{R}}^2\right)E_1 = -\left(k^2 - k_{\mathcal{L}}^2\right)E_2 \qquad E_3 = \frac{\Delta_k^2}{k_0^2}\left(E_1 - E_2\right) \tag{31}$$

Finally, the two eigenvectors in the circular polarization basis of the light $(\mathcal{R}\ ;\ \mathcal{L}\ ;\ \mathcal{Z})$ to lowest order in Δ_k^2 are:

$$k_{\mathcal{R}}\ :\ (\ 1\ ;\ 0\ ;\ 0\ ) + \frac{\Delta_k^2}{k_0^2}\left(\ 0\ ;\ -\frac{\tan(\theta)}{4}\ ;\ +1\ \right) \tag{32}$$

$$k_{\mathcal{L}}\ :\ (\ 0\ ;\ 1\ ;\ 0\ ) + \frac{\Delta_k^2}{k_0^2}\left(\ +\frac{\tan(\theta)}{4}\ ;\ 0\ ;\ -1\ \right) \tag{33}$$

This means that polarization eigenvector basis:

1. is very well approximated by the circular polarization basis of the light

2. has a small admixture of linear polarization

3. is slightly parallel to the propagation direction