Estimating the Size of Magnetic Fluctuations Due to Johnson Noise Currents

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1 Introduction

Both the Ra EDM Experiment and the Noble Gas Ice Project involve very sensitive measurements that require, among other things, very stable magnetic fields. One source of magnetic field instablility that could potentially limit the sensitivity of these experiments is the Johnson-Nyquist noise [1, 2] from conducting materials near the detection region. Thermal agitation (i.e. energy fluctuations) of the charge carriers inside conductors give rise to this electronic noise with a nearly frequency-independant spectral power density of:

$$\frac{dP_n}{d\nu} = 4kT \qquad \leftrightarrow \qquad \frac{dP_n}{d\omega} = \frac{2kT}{\pi} \tag{1}$$

where $k = 1.38\times10^{-23}$ J/K is the Boltzmann constant, T is the temperature in Kelvin, and $\omega = 2\pi\nu$. A derivation of this equation as well as a discussion of its frequency dependence is given in [3]. The relationship between Johnson noise and Shot noise is given in [4].

By noting that the power dissipated by a resistor is given $P = V^2/R = I^2R$, we can rewrite the noise spectrum in terms of the root mean square (RMS) voltage fluctuations across a resistor as:

$$\sqrt{V_n^2} \quad = \quad \sqrt{\frac{dV_n^2}{d\nu}(\Delta\nu)} \quad = \quad \sqrt{4kTR(\Delta\nu)} \tag{2}$$

or, alternatively, the RMS current noise traversing the resistor as:

$$\sqrt{I_n^2} \quad = \quad \sqrt{\frac{dI_n^2}{d\nu}(\Delta\nu)} \quad = \quad \sqrt{\frac{4kT(\Delta\nu)}{R}} \tag{3}$$

where R is the resistance and $\Delta\nu$ is the bandwidth. This current noise generates a magnetic field noise spectrum that, in general, depends on the geometry of & distance from the conductor and the frequency. In this document, we'll derive a set of equations that are subsequently used to estimate the effect the Johnson noise on the Ra EDM Experiment, see Sec. (3.1), and the Noble Gas Ice Project, see Sec. (3.2).

2 Magnetic Noise Spectrum

2.1 Inifinite Conducting Slab in the Dynamic Case

According to Eqn. (3), the current noise spectral density ($dI_n^2/d\nu$) is frequency independant. For sufficiently low frequencies, this results in magnetic noise that is also frequency independant. However, for high frequencies, the current noise dynamically generates compensating eddy currents that reduce the overall magnetic noise. Varpula & Poutanen [5] attempted to model the full frequency dependance of the magnetic noise by combining Eqn. (3) with Maxwell's equations. Based on semi-analytic calculations, they found an approximate expression for the magnetic noise density at a distance z from the closest surface of an infinite conducting plane with thickness $d = sz$:

$$\begin{aligned} \frac{dB_{n,z}^2}{d\nu} &= \left[\frac{\mu_0^2 kT}{8\pi\rho}\right]\left[\frac{s}{z(1+s)}\right]\left[\frac{1}{1+(\nu/\nu_c)^2}\right] = 2\left(\frac{dB_{n,x}^2}{d\nu}\right) = 2\left(\frac{dB_{n,y}^2}{d\nu}\right) \\ &= \left[\frac{38.9\ \mathrm{nG}}{\sqrt{\mathrm{Hz}}}\right]^2\left[\frac{T}{298\ \mathrm{K}}\right]\left[\frac{\rho_{\mathrm{Cu}}(298\ \mathrm{K})}{\rho(T)}\right]\left[\frac{1\ \mathrm{mm}}{z}\right]\left[\frac{s}{1+s}\right]\left[\frac{1}{1+(\nu/\nu_c)^2}\right] \end{aligned} \tag{4}$$

where $\sqrt{B_{n,q}^2}$ is the RMS magnetic noise in the q direction, the magnetic permeability of the plane is taken to be the magnetic constant $\mu_0 = 4\pi\times10^{-7}\ \mathrm{T\cdot m/A}$, ρ is the resistivity of the plane, $\rho_{\mathrm{Cu}}(298\ \mathrm{K}) = 1.71\times10^{-8}\ \Omega\cdot\mathrm{m}$ is the resistivity of Copper at room temperature, the x & y directions lie in the plane, the z direction is normal to the plane, and ν_c is the characteristic frequency is given by:

$$\nu_c = \frac{\pi\rho}{8\mu_0 z^2 s} = \left[\frac{5.34\ \mathrm{kHz}}{s}\right]\left[\frac{\rho(T)}{\rho_{\mathrm{Cu}}(298\ \mathrm{K})}\right]\left[\frac{1\ \mathrm{mm}}{z}\right]^2 \tag{5}$$

Munger [6] and Lee & Romalis [7] discuss this frequency dependence and its relationship to the magnetic permeablility μ of various materials when $\mu \neq \mu_0$. Based on symmetry considerations, Varpula & Poutanen also noted the following relationship among the different field noise compoments:

$$\frac{dB_{n,x}^2}{d\nu} = \frac{dB_{n,y}^2}{d\nu} = \frac{1}{2}\left(\frac{dB_{n,z}^2}{d\nu}\right) \tag{6}$$

As we'll show in the next section, this is not consistent with a numerical integration of the relevant equations.

We can calculate the flux noise in a loop that lies in the xy-plane but displaced a distance z from the surface of the conductor by multiplying the RMS magnetic field noise by the area of the loop a to give:

$$\begin{aligned} \frac{d\Phi_{n,z}^2}{d\nu} &= a^2\left(\frac{dB_{n,z}^2}{d\nu}\right) = \left[\frac{\mu_0^2 kTa^2}{8\pi\rho}\right]\left[\frac{s}{z(1+s)}\right]\left[\frac{1}{1+(\nu/\nu_c)^2}\right] \\ &= \left[\frac{0.188\cdot\Phi_0}{\sqrt{\mathrm{Hz}}}\right]^2\left[\frac{a}{1\ \mathrm{cm}^2}\right]^2\left[\frac{T}{298\ \mathrm{K}}\right]\left[\frac{\rho_{\mathrm{Cu}}(298\ \mathrm{K})}{\rho(T)}\right]\left[\frac{1\ \mathrm{mm}}{z}\right]\left[\frac{s}{1+s}\right]\left[\frac{1}{1+(\nu/\nu_c)^2}\right] \end{aligned} \tag{7}$$

where $\Phi_0 = 207\ \mathrm{nG}\cdot\mathrm{cm}^2$ is the magnetic flux quantum.

Since there is a frequency rolloff to the noise spectrum, the total noise integrated over an infinite bandwidth is finite and is given by:

$$\begin{aligned} \sqrt{B_{n,z}^2} &= \sqrt{\int_0^\infty \frac{dB_{n,z}^2}{d\nu}\,d\nu} = \sqrt{\frac{\pi\mu_0 kT}{128 z^3(1+s)}} \\ &= \frac{3.56\ \mu\mathrm{G}}{\sqrt{1+s}}\sqrt{\left[\frac{T}{298\ \mathrm{K}}\right]\left[\frac{1\ \mathrm{mm}}{z}\right]^3} \end{aligned} \tag{8}$$

$$\begin{aligned} \sqrt{\Phi_{n,z}^2} &= a\sqrt{B_{n,z}^2} = \sqrt{\frac{\pi\mu_0 kTa^2}{128 z^3(1+s)}} \\ &= \frac{17.2\cdot\Phi_0}{\sqrt{1+s}}\left[\frac{a}{1\ \mathrm{cm}^2}\right]\sqrt{\left[\frac{T}{298\ \mathrm{K}}\right]\left[\frac{1\ \mathrm{mm}}{z}\right]^3} \end{aligned} \tag{9}$$

It is interesting to note that the total integrated noise is independant of the resistivity of the material. Furthermore, by limiting the bandwidth to a few Hz (by integrating the signal over a few seconds), the noise can be reduced by at least two orders of magnitude.

An infinite plane conductor can be used to model a finite conductor when the distance z is much smaller than the thickness of the conductor (i.e. $s \gg 1$). In this case, the details of the spatial geometry of the conductor are only a first order correction ($1/s$) to the magnetic noise at low frequency (i.e. $\nu \ll \nu_c$) and s is interpreted as:

$$s = \frac{\text{characteristic size of the conductor}}{\text{characteristic distance from the surface of the conductor}} \tag{10}$$

On the other hand, the characteristic frequency ν_c is always inversely proportional to s, which implies that, at a fixed distance, the noise rolls off at a lower frequency for larger conductors.

2.2 Arbitrary Geometries in the Quasistatic Case

In general, it is quite difficult to follow the prescripton of Varpula & Poutanen for arbitrary geometries. However, as pointed out by Lamoreux [8], calculating the noise density at zero frequency (i.e. $\nu \ll \nu_c$) always provides a conservative upper limit for the noise density at all frequencies. In this case, called the quasistatic case, we ignore the effect of eddy currents and are able to directly apply the Biot-Savart Law to calculate the magnetic field from a steady state current distribution:

$$d\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\left[\frac{I d\vec{\ell} \times (\vec{r} - \vec{u})}{|\vec{r} - \vec{u}|^3}\right] \tag{11}$$

where $\vec{B}(\vec{r})$ is the magnetic field at the location $\vec{r}$, I is the current, and $d\vec{\ell}$ is the line element in the direction of the current at the location $\vec{u}$. This integral over $d\vec{\ell}$ is assumed to be zero for randomly fluctuating noise currents.

On the other hand, the RMS magnetic field is not expected to be zero and each component can be written as:

$$
\begin{aligned}
dB_3^2 &= \frac{\mu_0^2}{16\pi^2}\left[\frac{\{I_1 d\ell_1\,(r_2-u_2) - I_2 d\ell_2\,(r_1-u_1)\}^2}{|\vec{r}-\vec{u}|^6}\right] \\
&= \frac{\mu_0^2}{16\pi^2}\left[\frac{I_1^2(d\ell_1)^2\,(r_2-u_2)^2 + I_2^2(d\ell_2)^2\,(r_1-u_1)^2 - 2I_1I_2(d\ell_1)(d\ell_2)(r_1-u_1)(r_2-u_2)}{|\vec{r}-\vec{u}|^6}\right]
\end{aligned} \tag{12}
$$

where I_q is the current in the q direction and the subscripts $q = 1, 2, 3$ label the component of the vectors such that $\hat{1}\times\hat{2} = \hat{3}$. The randomly fluctuating noise currents in two different directions are assumed to be completely uncorrelated. Therefore, the cross term (i.e. I_1I_2) is assumed to integrate to zero and only the quadratic terms (i.e. I_1^2, I_2^2) survive. The field noise density can be written in terms of the current noise density, which, in the q direction, is given by:

$$
\frac{dI_{n,q}^2}{d\nu} = \frac{4kT}{R_q} = \frac{4kT}{\rho}\left(\frac{dA_q}{d\ell_q}\right) \tag{13}
$$

where R_q is the resistance in the q direction, $d\ell_q$ is the length in the q direction, and dA_q is the cross sectional area normal to the q direction. Plugging this into Eqn. (12) and dropping the cross terms (as argued before), we find that the q component of the field noise density is given by:

$$
\frac{dB_{n,q}^2}{d\nu} = \left(\frac{\mu_0^2 kT}{4\pi^2\rho}\right)\int\left|\frac{(\vec{r}-\vec{u})\times\hat{q}}{|\vec{r}-\vec{u}|^3}\right|^2 d^3u \propto \frac{TV}{\rho d^4} \tag{14}
$$

where $d^3u = (d\ell_x)(d\ell_y)(d\ell_z)$, V is the volume of the conductor, and d is the effective distance from the conductor. Note that as $V\to\infty$, such as in the case of an infinite plane conductor, $d^3\to\infty$ at the same rate and we rederive the result that the field noise scales as $1/\sqrt{d}$. Incidentally, a numerical integration of this equation suggests that, contrary to the conclusion of Varpula & Poutanen, the three components of the magnetic field noise due to an infinite plane are related by:

$$\left(\frac{dB_{n,x}^2}{d\nu}\right) = \left(\frac{dB_{n,y}^2}{d\nu}\right) \approx \frac{3}{2}\left(\frac{dB_{n,z}^2}{d\nu}\right) \tag{15}$$

A quick and dirty approach to calculating the flux noise density is to simply multiply Eqn. (14) by the square of the effective area of the pickup coil. For a more sophisticated estimate, we'll start by either representing the flux using the Biot-Savart Law or in terms of the magnetic vector potential $\vec{A}$:

$$\begin{aligned}\Phi &= \int \vec{B}(\vec{r})\cdot d\vec{a} = \frac{\mu_0}{4\pi}\int\oint \frac{Id\vec{\ell}\times(\vec{r}-\vec{u})\cdot d\vec{a}}{|\vec{r}-\vec{u}|^3} \\ &= \oint \vec{A}(\vec{r})\cdot d\vec{s} = \frac{\mu_0}{4\pi}\oint\oint \frac{Id\vec{\ell}\cdot d\vec{s}}{|\vec{r}-\vec{u}|}\end{aligned} \tag{16}$$

where $d\vec{a}$ is an element of area vector at a location $\vec{r}$ inside the loop or $d\vec{s}$ is the line element of the boundary that encloses the flux at a location $\vec{r}$. By following the same line of reasoning as before (i.e. terms involving the product of current components $I_j I_k$ integrate to zero unless $j = k$), we find that the flux noise density is given by:

$$\frac{d\Phi_n^2}{d\nu} = \left(\frac{\mu_0^2 kT}{4\pi^2\rho}\right)\int\left|\int \frac{(\vec{r}-\vec{u})\times d\vec{a}}{|\vec{r}-\vec{u}|^3}\right|^2 d^3u = \left(\frac{\mu_0^2 kT}{4\pi^2\rho}\right)\int\left|\oint \frac{d\vec{s}}{|\vec{r}-\vec{u}|}\right|^2 d^3u \propto \frac{TVa^2}{\rho d^4} \tag{17}$$

where a is the total area enclosed by the loop and d is the effective distance between the noise source and the loop. Note that as $V \to \infty$, such as in the case of an infinite plane conductor, $d^3 \to \infty$ at the same rate and we rederive the result that the flux noise scales as $a/\sqrt{d}$. On the other hand, as $a \to \infty$, such as when the coil is much larger than the conductor, $d^2 \to \infty$ at the same rate and we find that the flux noise scales as $\sqrt{V}$. Finally, we'll note the following useful scale factors:

$$\frac{\mu_0^2 kT}{4\pi^2\rho} = \left[\frac{9.81\ \mathrm{nG}}{\sqrt{\mathrm{Hz}}}\right]^2\left[\frac{T}{298\ \mathrm{K}}\right]\left[\frac{\rho_{\mathrm{Cu}}(298\ \mathrm{K})}{\rho(T)}\right]\cdot \mathrm{cm} = \left[\frac{0.0474\cdot\Phi_0}{\sqrt{\mathrm{Hz}}}\right]^2\left[\frac{T}{298\ \mathrm{K}}\right]\left[\frac{\rho_{\mathrm{Cu}}(298\ \mathrm{K})}{\rho(T)}\right]\cdot \mathrm{cm}^{-3} \tag{18}$$

3 Numerical Results

3.1 Ra EDM Experiment

The Radium EDM Experiment is a search for the permanent electric dipole moment (EDM) of the Ra-225 nucleus. Laser-cooled & trapped Ra atoms are spin-polarized by optical pumping transverse to a small ($\approx 10^{-2}$ G) bias DC magnetic field. As a consequence, the Ra atoms precess freely about the magnetic field at the Larmor frequency ($\approx$10 Hz). The experimental observable is the small shift in this Larmor frequency which could be produced by the coupling of a nonzero EDM to a strong ($\approx 10^2$ kV/cm) DC electric field. This shift can be isolated by reversing the sign of the electric field relative to the magnetic field.

In order to perform a competitive EDM experiment ($|d| < 10^{-26}\ e \cdot \text{cm}$), we will achieve an integrated relative statistical uncertainty on the frequency shift of 0.1 ppm. This will be accomplished by 10^4 frequency shift measurements, each with a relative statistical uncertainty of $(0.1\ \text{ppm})\sqrt{10^4} = 10$ ppm [9]. The statistical precision of each measurement of the frequency shift improves linearly with (1) the number of atoms N (for small N) and (2) the observation time T [10]. In order to achieve this statistical precision, we have estimated that each measurement should consist of $N \approx 10^4$ atoms with $T_{\text{obs}} \approx 10^2$ seconds [11].

The main source of magnetic Johnson noise are the two electrodes used to generate the electric field. These electrodes are separated by 2 mm, while the Ra atoms are located directly in the center between them. Combining this geometry with the dimensions depicted in Fig. (1), we obtain the following results via numerical integration with $4 \times (2 \times 100 + 1) \times (64^2)$ volume elements per electrode:

$$\int \left| \frac{(\vec{r} - \vec{u}) \times \hat{x}}{|\vec{r} - \vec{u}|^3} \right|^2 d^3u = \int \left| \frac{(\vec{r} - \vec{u}) \times \hat{y}}{|\vec{r} - \vec{u}|^3} \right|^2 d^3u = \frac{42.4}{\text{cm}} \qquad \int \left| \frac{(\vec{r} - \vec{u}) \times \hat{z}}{|\vec{r} - \vec{u}|^3} \right|^2 d^3u = \frac{25.7}{\text{cm}} \quad (19)$$

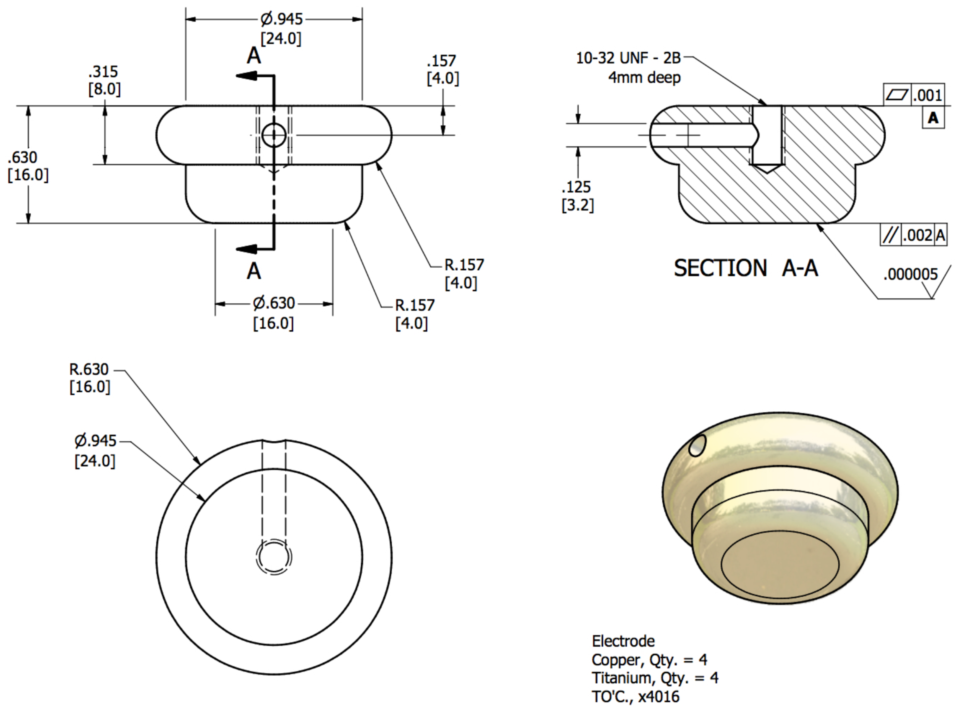

Figure 1: Electrode Dimensions. The electrodes are made from copper [12].

Assuming the electrodes are made of copper & are held at room temperature ($T = 298$ K) and the atoms are observed for $T_{\mathrm{obs}} = 10^2$ sec ($\Delta\nu = 1/T_{\mathrm{obs}} = 0.01$ Hz), we find:

$$\sqrt{B^2_{n,x}} = \sqrt{B^2_{n,y}} = \frac{63.9\text{ nG}}{\sqrt{\text{Hz}}}\sqrt{\frac{1}{100\text{ s}}} = 6.39\,\text{nG} \qquad \sqrt{B^2_{n,z}} = \frac{49.7\text{ nG}}{\sqrt{\text{Hz}}}\sqrt{\frac{1}{100\text{ s}}} = 4.97\,\text{nG} \quad (20)$$

This corresponds to fluctuations of the bias DC field ($\approx 10^{-2}$ G in the z-direction) of 0.5 ppm, which is well below the desired per-measurement statiscal precision of 10 ppm. Since $\rho_{\mathrm{Ti}}/\rho_{\mathrm{Cu}} = 23.2$, the relative frequency uncertainty due to Titantium electrodes of the same geometry is about 0.1 ppm.

We can also perform this calculation for a austentic stainless steel tube with a length of 160 cm, radius of 3.5 cm, and a wall thickness of 5 mm. At the center of the tube, the spatial integrals ($(2 \times 800 + 1) \times 35^2$) for the magnetic noise are 0.294/cm in the radial direction and 0.353/cm along the axis of the tube. For this geometry, the magnetic noise, using $\rho_{\mathrm{SS}}/\rho_{\mathrm{Cu}} = 44.1$, is 0.8 nG/$\sqrt{\text{Hz}}$, which corresponds to 8 ppb after 10^2 sec.

3.2 Nobel Gas Ice Project

One of the near-term goals of the Noble Gas Ice Project is to demonstrate the viability of matrix-isolated diamagnetic atoms with nonzero nuclear spins as a "next-generation" tool for nuclear EDM searches. The major potential benefit of this approach would be the increased statistical sensitivity to the EDM signal due to the orders of magnitude higher number of atoms in the sample relative to the sample size of traditionally optically-trapped atomic species. Once polarized, the nuclear spins are expected to retain their nonthermal polarization for a very long time, which is ideal for an EDM search. Towards this end, we have chosen to demonstrate the proof-of-principle of this approach with Ytterbium (Yb) atoms embedded in a solid-Neon (s-Ne) matrix.

Our goal is to optically pump Yb atoms while embedded in the s-Ne matrix and detect the resulting magnetization due to the nuclear spin polarization of Yb-171 in natural abundance Yb. Our planned detection technique takes advantage of the coupling between this magnetization and a "pickup" coil as measured by a SQUID. In a previous note, [13], assuming a natural abundance Yb to Ne ratio of $[\mathrm{Yb}]/[\mathrm{Ne}] = 10^{-5}$, a 100% Yb-171 nuclear polarization, a 1 cm diameter sample with a 30 μm thickness, and a 1 cm diameter coil, we calculated that the flux measured by a single loop coil 2 mm from the sample would be about 15.1 mΦ_0. HERE A potentially major source of noise that may overwhelm this small signal is the magnetic Johnson from the metal components

A Noise, Observation Time,

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