A Guide to Radiative Corrections

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Abstract

A guide to radcor.

Contents

1 Scattering Amplitude and Cross Sections

The Born cross section for inclusive electron scattering is given by the lowest order feynman diagram. The differences between the Born cross section and the measured cross section are radiative corrections. These corrections are due to two effects:

1. higher order corrections due to next to leading order feynman diagrams
2. energy loss by the electron

Higher order corrections are due to the loop diagrams. The vertex loop is composed of:

1. a finite term that results in the anomalous magnetic moment of the electron
2. an ultraviolet divergence that is cancelled by the electron self energy diagrams
3. an infrared divergence that is cancelled by the electron self energy diagrams
4. an infrared divergence that is cancelled by the low energy part of internal bremsstrahlung

The electron self energy diagram is composed of:

1. an ultraviolet divergence that is cancelled by the vertex diagram
2. an infrared divergence that is cancelled by the vertex diagram
3. a lograthmic correction to the electron mass due to renomarlization

The vacuum diagram is composed of:

1. a lograthmic correction to the electromagnetic coupling constant due to renomarlization

The vertex, electron self energy, and vacuum diagrams add toghether to give the scattering amplitude. The cross section is proportional to the modulus squared of the scattering amplitude. A term from the vertex diagram is infrared divergent. This is cancelled by a corresponding term from the internal bremsstrahlung cross section. Note that this process is a distinct one. Specifically, one calculates the sum of the mod squares to find the total cross section for electron scattering and internal bremsstrahlung as opposed to finding the mod square of the sum of the amplitudes for the two processes. Any physical detector has a non zero energy resolution; therefore the two processes are indistinguisable and only sum is observable. This is reassuring because only the sum is finite.

$$\begin{aligned}
\mathcal{M}_{\text{full}} &= \alpha\mathcal{M}_{\text{Born}} + \alpha^2\left(\mathcal{M}_{\text{vertex}} + \mathcal{M}_{\text{vacuum}} + \mathcal{M}_{\text{self}}\right) + \cdots && (1)\\
d\sigma_{\text{full}} &\propto \left|\mathcal{M}_{\text{full}}\right|^2 \approx \alpha^2\left[\mathcal{M}^2_{\text{Born}} + 2\alpha\mathcal{M}_{\text{born}}\Re\left(\mathcal{M}_{\text{vertex}} + \mathcal{M}_{\text{vacuum}} + \mathcal{M}_{\text{self}}\right)\right] + \cdots && (2)\\
\left[\frac{d^2\sigma}{dEd\Omega}\right]_{\text{full}} &= \left(1 + \delta_{\text{HO}} + \mathbf{IR} + \cdots\right)\left[\frac{d^2\sigma}{dEd\Omega}\right]_{\text{Born}} && (3)\\
\left[\frac{d^2\sigma}{dEd\Omega}\right]_{\text{IB}} &= \left[\frac{d^2\sigma}{dEd\Omega}\right]_{\text{IB}<} + \left[\frac{d^2\sigma}{dEd\Omega}\right]_{\text{IB}>} && (4)\\
\left[\frac{d^2\sigma}{dEd\Omega}\right]_{\text{IB}<} &= \left(-\mathbf{IR}\right)\left[\frac{d^2\sigma}{dEd\Omega}\right]_{\text{Born}} && (5)\\
\left[\frac{d^2\sigma}{dEd\Omega}\right]_{\text{EM}} &= \left[\frac{d^2\sigma}{dEd\Omega}\right]_{\text{full}} + \left[\frac{d^2\sigma}{dEd\Omega}\right]_{\text{IB}} = \left(1 + \delta_{\text{HO}} + \cdots\right)\left[\frac{d^2\sigma}{dEd\Omega}\right]_{\text{Born}} + \left[\frac{d^2\sigma}{dEd\Omega}\right]_{\text{IB}>} && (6)
\end{aligned}$$

2 Internal Bremsstrahlung

The internal bremsstrahlung differentional cross section for a discrete state is an integral over all angles for the emitted photon:

$$\left[\frac{d^2\sigma}{dEd\Omega}\right]_{\text{IntB}(\omega>\omega_0)} = \int B_{\mu\nu}T_{\mu\nu}\ d\Omega_k = \int a_1W_1 + a_2W_2\ d\Omega_k \tag{7}$$

where $B_{\mu\nu}$ is the internal bremsstrahlung tensor and $T_{\mu\nu}$ is the leptonic tensor which contains the structure functions $W_{1,2}$. If the continuum is taken to be a spectrum of discrete states, then the internal bremsstrahlung differentional cross section is just a sum over all states:

$$\left[\frac{d^2\sigma}{dEd\Omega}\right]_{\text{IntB}(\omega>\omega_0)} = \int\int a_1W_1 + a_2W_2\ d\Omega_k\ dM_f^2 = \int\int a_1'W_1 + a_2'W_2\ d\Omega_k\ d\omega \tag{8}$$

For elastic scattering, the internal bremsstrahlung cross section is obtained via a numerical intergration of the form factors over all emitted photon angles. This is equation (B5) of MT69 or (A.24) of TSAI71 or A24 of STEIN. For the continous spectrum:

1. the angle peaking approximation is made to simplify the angular integration. It take advantage of the fact that emission of a photon is most likely to happen in the direction of the electron momentum. Therefore this approximation results in two terms, one due to emission in the direction of the incident electron and one for emission of the photon in the direction of the scattered electron.
2. the structure functions in the integrand are replaced by the unradiated cross section. The substitution is not perfect and the "residual" terms are dropped.

3. the ω integral is rewritten in terms of E_s and E_p.

"Explicitly" in equation form:

$$\left[\frac{d^2\sigma}{dEd\Omega}\right]_{\mathrm{IntB}(\omega>\omega_0)} \approx \left[\int a_1''W_1 + a_2''W_2 \; d\omega\right]_{\omega\|s} + \left[\int a_1'''W_1 + a_2'''W_2 \; d\omega\right]_{\omega\|p} \tag{9}$$

$$\approx \int I_{\mathrm{IntB}} \left[\frac{d^2\sigma}{dEd\Omega}\right]_{\mathrm{eff}} dE_s' + \int I_{\mathrm{IntB}} \left[\frac{d^2\sigma}{dEd\Omega}\right]_{\mathrm{eff}} dE_p' \tag{10}$$

where I_{IntB} is the probability of internal bremsstrahlung for given photon, incident electron, and scattered electron energies. This function is essentially the product of the effective radiation thickness and the internal bremstrahlung spectrum and some other stuff:

$$[bt\phi(v)]_{\mathrm{IntB}} = \frac{\alpha}{\pi}\left[(1-v)\left\{\log\left(\frac{Q^2}{m^2}\right)-1\right\} + \frac{v^2}{2}\log\left(\frac{4E}{m^2}\right)\right] \text{ (Hand)} \tag{11}$$

$$= \frac{\alpha}{\pi}\left[\log\left(\frac{Q^2}{m^2}\right)-1\right]\left[1-v+\frac{1}{2}v^2\right] \text{ (Allton \& Bjorken)} \tag{12}$$

$$= \frac{\alpha}{\pi}\left[\left(1-v+\frac{v^2}{2}\right)\log\left(\frac{Q^2}{m^2}\right)-1+v\right] \text{ (Mo \& Tsai)} \tag{13}$$

$$= \frac{\alpha}{\pi}\left[\log\left(\frac{Q^2}{m^2}\right)-1\right]\left[1-v+\frac{3}{4}v^2\right] \text{ (Equivalent Radiators)} \tag{14}$$

needs work also say something about multiple photon correction

3 External Bremsstrahlung and Collisional Loss

Electrons lose energy passing through materials either by ionizing collisions with atomic electrons or via bremsstrahlung in Coulombic field within atoms. The amount of energy lost is a statistcal quantity. If energy is lost before the nuclear scattering takes place, then the incident energy is lower than the beam energy. If energy is lost after the nuclear scattering, then the momentum of the detected electron is lower than it's value immediately after scattering. Therefore the raw experimental cross section for a given set of kinematics is actually a convolution over a spread of kinematics:

$$\sigma_{\mathrm{exp}} = \int_0^T \int_{p_{\mathrm{mom}}}^{E_{\mathrm{beam}}} \int_{p_{\mathrm{mom}}}^{E_s} I(E_{\mathrm{beam}}, E_s, t)\sigma_{\mathrm{tot}}(E_s, E_p)I(E_p, p_{\mathrm{mom}}, T-t)\; dE_p\; dE_s\; \frac{dt}{T} \tag{15}$$

where $I(E_i, E_f, t)$ is the probability of losing $E_f - E_i$ amount of energy after travelling an amount t in radiation length and σ_{tot} includes internal bremsstrahlung and higher order corrections. In the limit of an infinitesimally thin target, we retrieve the "unradiated" cross section:

$$\lim_{T\to 0}\int_0^T \int_{p_{\mathrm{mom}}}^{E_{\mathrm{beam}}} \int_{p_{\mathrm{mom}}}^{E_s} I(E_{\mathrm{beam}}, E_s, t)\sigma_{\mathrm{tot}}(E_s, E_p)I(E_p, p_{\mathrm{mom}}, T-t)\; dE_p\; dE_s\; \frac{dt}{T} = \sigma_{\mathrm{tot}}(E_{\mathrm{beam}}, p_{\mathrm{mom}}) \tag{16}$$

To simplify this expression, TSAI71 makes the following arguments:

1. Use a simpler form of σ_{tot} to reduce things and then plug back the correct form of σ_{tot}. The simple form of σ_{tot} is σ_{un}.

2. Treat only the bremsstrahlung part of I. Add the collisional part of I to the reduced form at the very end by analogy to the bremsstrahlung part.

3. Calculate the integral for a discrete level. This means that the cross section will have an energy delta function which reduces the area integral into a line integral for constant missing mass.

4. Make the "energy" peaking approximation. This appoximation reduces the aforementioned line integral to the cross sections at the endpoints of the line. This approximation physically means that the most of the energy is lost to a single photon before or after scattering.

5. Subtract out the effect of all discrete levels until the contiuum pion threshold limit is reached.

6. Treat the contiunuum as a series of closely spaced discrete levels and sum over all of them. In other words, integrate over all missiing mass from the continuum to the point at which the cross section is being measured. iThe is equivalent to intergrating over one strip with at fixed E_p and another strip over fixed E_s. This result is the reduced form of the area integral.

7. Put back the correct form of σ_{tot} and add the contibution from collisional loss.

8. Shift the energies before and after scattering by an amount equal to the most probable collisional energy loss.

The form of I is obtained by making the following argument:

1. Eyges derives a form for I when the beam energy is small. By comparing the form of I to the form of the bremsstrahlung spectrum, one finds there is an extra multiplicative term. It is assumed that this extra term accounts for multiple photons or energy loss events.

2. The problem is that at high beam energies, the Eyges bremsstrahlung spectrum is not correct. Therefore TSAI simply substitutes the a more reasooable form of the bremsstrahlung spectrum at high energies keeping the form of the multple photon correction.

4 Inital and Final Electron Energies

We'll assume that $\hbar = c = 1$ except where noted. The electron is assumed to be very relativistic $\beta \simeq 1$, consequently $E \simeq p$ unless otherwise noted. The incident electron energy E_s and final electron energy E_p are related to the beam energy and spectrometer momentum by:

$$E_s = E_{\text{beam}} - \Delta_s \tag{17}$$

$$E_p = p_{\text{mom}} + \Delta_p \tag{18}$$

where $\Delta_{s,p}$ is the enegy loss due to ionizing collisions. For a pure element, the energy loss per unit density per unit thickness is given by:

$$\left[\frac{\Delta}{\rho x}\right] = \left[\frac{\xi}{\rho x}\right]\left[2\log\left(\frac{m_e c^2}{I_{\text{BB}}}\right) + \log\left(\gamma^2 - 1\right) - \delta(X) + g\right] \tag{19}$$

$$\left[\frac{\xi}{\rho x}\right] = \frac{Za}{A\beta^2} \tag{20}$$

$$a = 2\pi N_A r_e^2 m_e c^2 = 0.15353747 \text{ MeV} \cdot \frac{\text{cm}^2}{\text{mol}} \tag{21}$$

$$\beta = \frac{v}{c} = \frac{pc}{E} \tag{22}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{E}{m_{\text{e}} c^2} \tag{23}$$

where Z & A are the element's atomic number and weight (in g/mol), p & E are the electron's momentum and energy, ρ & x are the element's mass density and thickness, I_{BB} is the mean excitation potential of the material, $\delta(X)$ is the density correction, and g depends on whether one is interested in the mean loss or most probable loss. The density correction δ is given by [**?**, **?**]:

$$\delta(X) = \left\{\begin{array}{ll} \delta\left(X_0^\delta\right) \times 10^{2\left(X - X_0^\delta\right)} & X \leq X_0^\delta \\ 4.6052X + C_\delta + a_\delta\left(X_1^\delta - X\right)^{m_\delta} & X_0^\delta < X < X_1^\delta \\ 4.6052X + C_\delta & X > X_1^\delta \end{array}\right\} \tag{24}$$

$$X = \log_{10}(\beta\gamma) \tag{25}$$

where C_δ, X_0^δ, X_1^δ, a_δ, and m_δ depend on the element. For the mean energy loss, g is given by:

$$\begin{aligned}
\bar{g} &= \log(\gamma - 1) - F(\gamma) \qquad (26)\\
F(\gamma) &= \left[1 + \frac{2}{\gamma} - \frac{1}{\gamma^2}\right]\log(2) - \frac{1}{8}\left[1 - \frac{1}{\gamma}\right]^2 - \frac{1}{\gamma^2}\\
&\approx \log(2) - \frac{1}{8} = 0.568 \text{ for } \gamma \gg 1
\end{aligned}$$

and for the most probable energy loss, g is given by:

$$g_{\mathrm{mp}} = \log\left[\frac{2\gamma^2\xi}{m_e c^2}\right] - \beta^2 + 0.198 \qquad (27)$$

For a composite material, the energy loss due to ionization is given by:

$$\Delta = \left(\sum_k \left[\frac{\Delta}{\rho x}\right]_k w_k\right)\rho x \qquad (28)$$

$$\xi_k = \left[\frac{\xi}{\rho x}\right]_k w_k \rho x \qquad (29)$$

where w_k is the fraction by weight of element k in the material. For a sequence of materials located one after another, the energy loss due to ionization is given by:

$$\Delta = \sum_j \Delta_j = \sum_j \left(\sum_k \left[\frac{\Delta}{\rho x}\right]_k w_k^j\right)\rho_j x_j \qquad (30)$$

where w_k^j is the fraction by weight of the k-th element in the j-th material. TSAI71 argues that we should use the *most probable* energy loss, so that is what we will use.

5 Elastic Tail

In the formalism of TSAI71 and Stein, the elastic tail is a sum of three parts as implemented in the fortran program **ROSETAIL.F**:

$$\left[\frac{d^2\sigma}{dEd\Omega}\right]_{\mathrm{elastic}} = \left(R_t \cdot \left[\frac{d^2\sigma}{dEd\Omega}\right]_{\mathrm{InB}} + \left[\frac{d^2\sigma}{dEd\Omega}\right]_{\mathrm{ExB}} + \left[\frac{d^2\sigma}{dEd\Omega}\right]_{\mathrm{coll}}\right) \cdot F_{\mathrm{soft}} \qquad (31)$$

where R_t accounts for radiation emitted by the target nucleus and F_{soft} accounts for the radiation of multiple photons. It's not clear if this term should also be used to "correct" the part of the tail due to collisional loss, but we'll leave it as it is. The target radiation correction neglecting the electron mass is given by Miller 72:

$$R_t = 1 + \frac{\alpha}{\pi b t_r}\left[\left(\frac{1+2\tau}{2\tau\sqrt{1+1/\tau}}\right)\log\left(1 + 2\tau + 2\tau\sqrt{1+1/\tau}\right) - 1 + 2\log(\eta_s)\right] \qquad (32)$$

and the multple photon correction is given by

$$F_{\mathrm{soft}} = \left[v_s^{bt_b+bt_r}\right] \cdot \left[v_p^{bt_a+bt_r}\right] \qquad (33)$$

where the kinematic factors are:

$$\eta_s = 1 + \frac{2E_s}{M_T}\sin^2\left(\frac{\theta}{2}\right) \qquad (34)$$

$$\eta_p = 1 - \frac{2E_p}{M_T}\sin^2\left(\frac{\theta}{2}\right) \qquad (35)$$

$$Q^2 = 4E_sE_p\sin^2\left(\frac{\theta}{2}\right) \qquad (36)$$

$$\tau = \frac{Q^2}{4M_T^2} \qquad (37)$$

and the emitted photon energy is ω:

$$\begin{aligned} \omega_s &= E_s - E_p/\eta_p && (38) \\ \omega_p &= E_s/\eta_s - E_p && (39) \\ v_s &= \frac{\omega_s}{E_s} && (40) \\ v_p &= \frac{\omega_p}{E_p + \omega_p} && (41) \end{aligned}$$

where v is the emitted photon energy as a fraction of the electron energy. The external radiation lengths before and after the target are t_b & t_a and with different definitions of b:

$$\begin{aligned} L(Z) &= \log\left(184.15 Z^{-\frac{1}{3}}\right) - f(Z\alpha) \approx \log\left(191 Z^{-\frac{1}{3}}\right) - 1.2(Z\alpha)^2 \approx \log\left(183 Z^{-\frac{1}{3}}\right) && (42) \\ f(z) &= z^2 \sum_{\nu=1}^{\infty} \frac{1}{\nu\left(\nu^2 + z^2\right)} \approx z^2 \left[\frac{1}{1+z^2} + 0.20206 - 0.0369 z^2 + 0.0083 z^4 - 0.002 z^6\right] && (43) \\ L'(Z) &= \log\left(1194 Z^{-\frac{2}{3}}\right) \approx \log\left(1440 Z^{-\frac{2}{3}}\right) && (44) \\ b &= \frac{4}{3}\left[1 + \frac{1}{12}\left(\frac{Z+1}{ZL + L'}\right)\right] = \frac{4}{3} + \frac{1}{9}\left(\frac{Z+1}{ZL+L'}\right) \approx \frac{4}{3} && (45) \end{aligned}$$

The early papers did not take into account the Coulomb correction $f(z)$. In addition the older fomula for the second radiation integral $L'(Z)$ has the incorrect factor (1440) due to a mislabled graph in the old Wheeler and Lamb paper. Ultimately we'll simply use $b = 4/3$ because:

1. the bremsstrahlung cross section in the complete screening limit has an additional small term
2. the radiation length should also has a small additional term
3. to lowest order, these two small terms cancel
4. the moral of the story is that this small term should be included in *both* the bremsstrahlung spectrum and the radiation length or *in neither one*

The form factor modifier is:

$$\begin{aligned} \delta_{\text{vertex}} &= \frac{2\alpha}{\pi}\left[-1 + \frac{3}{4}\log\left(\frac{Q^2}{m^2}\right)\right] && (46) \\ \delta_{\text{vacuum}} &= \frac{2\alpha}{\pi}\left[-\frac{5}{9} + \frac{1}{3}\log\left(\frac{Q^2}{m^2}\right)\right] && (47) \\ \delta_{\text{schwinger}} &= \frac{\alpha}{\pi}\left[\frac{\pi^2}{6} - \Phi\left(\cos^2\left(\frac{\theta}{2}\right)\right)\right] && (48) \\ \delta_0 &= \frac{\alpha}{\pi}\left[\Phi\left(\frac{E_s - E_p}{E_s}\right) + \Phi\left(\frac{E_p - E_s}{E_p}\right)\right] \approx -\frac{\alpha}{2\pi}\log^2\left(\frac{E_s}{E_p}\right) && (49) \\ \delta_{\text{HO}} &= \delta_{\text{vertex}} + \delta_{\text{vacuum}} + \delta_{\text{schwinger}} + \delta_0 && (50) \\ \bar{F}(Q^2) &= \frac{\exp\left(\delta_{\text{HO}}\right)}{\Gamma(1+bt_b)\cdot\Gamma(1+bt_a)} \approx 1 + \delta_{\text{HO}} + 0.5772\left(bt_a + bt_b\right) && (51) \end{aligned}$$

where the Schwinger term has a simple form because $E_s \gg E_p$ is *not* true in the lab reference frame. It's not clear if the higher order δ factors should be exponentiated, but it has not practical difference since δ_{HO} is small. The gamma functions are due to the normalization of the external bremsstrahlung straggling fucntion. In principle the correct normalization is a product of two gamma functions, but there is little practical difference because bt is so small.

The internal bremsstrahlung process involves the following quantities in the lab frame of reference:

1. $s = (E_s, \vec{s})$, the incoming electron

2. $p = (E_p, \vec{p})$, the outcoming electron

3. $k = (\omega, \vec{k})$, the emitted photon

4. $p_i = (M_T, 0)$, the intial state of the target nucleus

5. $p_f = s + p_i - p - k$, the final state of the target nucleus

6. $u = (u_0, \vec{u}) = s + p_i - p = p_f + k$, the final state of the target nucleus had there been no radiation

7. $\hat{u}$ defines the z-axis of the coordinate system

8. $\vec{u}$, $\vec{s}$, and $\vec{p}$ all lie in the xz-plane

9. θ_s is the angle between $\vec{s}$ and $\vec{u}$

10. θ_p is the angle between $\vec{p}$ and $\vec{u}$

11. $\theta = \theta_p - \theta_s$ is the angle between $\vec{s}$ and $\vec{p}$ or equivalently the scattering angle in the lab frame

12. θ_k is the angle between $\vec{k}$ and $\vec{u}$

The exact calculation of elastic internal bremsstrahlung:

$$\left[\frac{d^2\sigma}{dEd\Omega}\right]^{\text{ex}}_{\text{InB}} = \frac{\alpha^3 E_p}{2\pi E_s}\int_{-1}^{+1} d\left(\cos(\theta_k)\right)\frac{4\omega^2 M_T}{q^4\left(u^2 - M_T^2\right)} \times \left[W_1(q^2)\left(A_s + A_p\right) + W_2(q^2)\left(B_s + B_p\right)\right]\bar{F}(q^2) \tag{52}$$

$$A_s = \frac{a_s m^2}{x_s^3}\left(2m^2 + q^2\right) + 2 + \frac{4\nu}{x_s}(s\cdot p)\left[(s\cdot p) - 2m^2\right] + \left[\frac{2(s\cdot p) + 2m^2 - q^2}{x_s}\right] \tag{53}$$

$$A_p = \frac{a_p m^2}{x_p^3}\left(2m^2 + q^2\right) + 2 - \frac{4\nu}{x_p}(s\cdot p)\left[(s\cdot p) - 2m^2\right] - \left[\frac{2(s\cdot p) + 2m^2 - q^2}{x_p}\right] \tag{54}$$

$$\begin{aligned} B_s = & -\frac{a_s m^2}{x_s^3}\left[2E_s\left(E_p + \omega\right) + \frac{q^2}{2}\right] - 1 + \frac{2\nu}{x_s}\left\{m^2\left[(s\cdot p) - \omega^2\right] + (s\cdot p)\left[2E_sE_p - (s\cdot p) + \omega\left(E_s - E_p\right)\right]\right\} \\ & + \left[\frac{2\left(E_sE_p + E_s\omega + E_p^2\right) + q^2/2 - (s\cdot p) - m^2}{x_s}\right] \end{aligned} \tag{55}$$

$$\begin{aligned} B_p = & -\frac{a_p m^2}{x_p^3}\left[2E_p\left(E_s - \omega\right) + \frac{q^2}{2}\right] - 1 - \frac{2\nu}{x_p}\left\{m^2\left[(s\cdot p) - \omega^2\right] + (s\cdot p)\left[2E_sE_p - (s\cdot p) + \omega\left(E_s - E_p\right)\right]\right\} \\ & - \left[\frac{2\left(E_sE_p - E_p\omega + E_s^2\right) + q^2/2 - (s\cdot p) - m^2}{x_p}\right] \end{aligned} \tag{56}$$

where A_s & B_s and A_p & B_p correspond to photon emission before and after scattering from the target nucleus respectively. The following electron kinematic factors include the small electron mass:

$$|\vec{s}| = \sqrt{E_s^2 - m^2} \tag{57}$$

$$|\vec{p}| = \sqrt{E_p^2 - m^2} \tag{58}$$

$$(s\cdot p) = E_sE_p - |\vec{s}||\vec{p}|\cos(\theta) \tag{59}$$

$$u^2 = 2m^2 + M_T^2 - 2(s\cdot p) + 2M_T\left(E_s - E_p\right) \tag{60}$$

$$u_0 = E_s + M_T - E_p \tag{61}$$

$$|\vec{u}| = \sqrt{u_0^2 - u^2} \tag{62}$$

$$\omega = \frac{u^2 - M_T^2}{2\left[u_0 - |\vec{u}|\cos(\theta_k)\right]} \tag{63}$$

$$q^2 = 2m^2 - 2(s\cdot p) - 2\omega\left(E_s - E_p\right) + 2\omega|\vec{u}|\cos(\theta_k) \tag{64}$$

$$\cos(\theta_s) = \frac{|\vec{s}| - |\vec{p}|\cos(\theta)}{|\vec{u}|} \tag{65}$$

$$\cos(\theta_p) = \frac{|\vec{s}|\cos(\theta) - |\vec{p}|}{|\vec{u}|} \tag{66}$$

These are kinematic factors that correspond to the emitted photon:

$$a_s = \omega\left[E_s - |\vec{s}|\cos(\theta_s)\cos(\theta_k)\right] \tag{67}$$

$$a_p = \omega\left[E_p - |\vec{p}|\cos(\theta_p)\cos(\theta_k)\right] \tag{68}$$

$$\nu = \frac{1}{a_s - a_p} \tag{69}$$

$$b_s = b_p = -\omega|\vec{s}|\sqrt{1-\cos^2(\theta_s)}\sqrt{1-\cos^2(\theta_k)} = -\omega|\vec{p}|\sqrt{1-\cos^2(\theta_p)}\sqrt{1-\cos^2(\theta_k)} \tag{70}$$

$$x_s = \sqrt{a_s^2 - b_s^2} \tag{71}$$

$$x_p = \sqrt{a_p^2 - b_p^2} \tag{72}$$

The elastic cross section and form factors are defined as:

$$\left[\frac{d\sigma(E)}{d\Omega}\right]_{\rm el} = \eta\left[\frac{4\alpha E\cos\left(\frac{\theta}{2}\right)}{\eta Q^2}\right]^2\left[W_2\left(Q^2\right) + 2\tan^2\left(\frac{\theta}{2}\right)W_1\left(Q^2\right)\right] \tag{73}$$

$$\eta = 1 + \frac{2E}{M_T}\sin^2\left(\frac{\theta}{2}\right) \tag{74}$$

$$Q^2 = 2EM_T\left(1 - \frac{1}{\eta}\right) \tag{75}$$

$$\tau = \frac{Q^2}{4M_T^2} \tag{76}$$

$$W_1\left(Q^2\right) = \tau\left[G_M\left(Q^2\right)\right]^2 \tag{77}$$

$$W_2\left(Q^2\right) = \frac{\left[G_E\left(Q^2\right)\right]^2 + \tau\left[G_M\left(Q^2\right)\right]^2}{1+\tau} \tag{78}$$

The peaking approximation for the elastic internal is:

$$\left[\frac{d^2\sigma}{dE_p d\Omega}\right]^{\rm pk}_{\rm InB} = \left[\frac{d^2\sigma}{dE_p d\Omega}\right]^{\rm k\|s}_{\rm InB} + \left[\frac{d^2\sigma}{dE_p d\Omega}\right]^{\rm k\|p}_{\rm InB} \tag{79}$$

$$\left[\frac{d^2\sigma}{dE_p d\Omega}\right]^{\rm k\|s}_{\rm InB} = \left[\frac{\eta_s - (\eta_s - 1)\,v_s}{\eta_p}\right]\left[\frac{bt_r\phi(v_s)}{\omega_s}\right]\bar{F}\left(Q_s^2\right)\left[\frac{d\sigma(E_s - \omega_s)}{d\Omega}\right]_{\rm el} \tag{80}$$

$$\left[\frac{d^2\sigma}{dE_p d\Omega}\right]^{\rm k\|p}_{\rm InB} = \left[\frac{bt_r\phi(v_p)}{\omega_p}\right]\bar{F}\left(Q_p^2\right)\left[\frac{d\sigma(E_s)}{d\Omega}\right]_{\rm el} \tag{81}$$

$$Q_s^2 = 2\left(E_s - \omega_s\right)M_T\left(1 - \frac{1}{\eta_s - (\eta_s - 1)\,v_s}\right) \tag{82}$$

$$Q_p^2 = 2E_sM_T\left(1 - \frac{1}{\eta_s}\right) \tag{83}$$

where $bt_r\phi(v)$ is the product of effective internal bremsstrahlung radiation length and effective bremmstrhalung spectrum. MT69 list four different variations of the peaking approximation:

$$\left[bt_r\phi(v)\right]_{\rm InB} = \frac{\alpha}{\pi}\left[(1-v)\left\{\log\left(\frac{Q^2}{m^2}\right) - 1\right\} + \frac{v^2}{2}\log\left(\frac{4E}{m^2}\right)\right] \text{ (Hand)} \tag{84}$$

$$= \frac{\alpha}{\pi}\left[\log\left(\frac{Q^2}{m^2}\right) - 1\right]\left[1 - v + \frac{1}{2}v^2\right] \text{ (Allton \& Bjorken)} \tag{85}$$

$$= \frac{\alpha}{\pi}\left[\left(1-v+\frac{v^2}{2}\right)\log\left(\frac{Q^2}{m^2}\right)-1+v\right] \text{ (Mo \& Tsai)} \quad (86)$$

$$= \frac{\alpha}{\pi}\left[\log\left(\frac{Q^2}{m^2}\right)-1\right]\left[1-v+\frac{3}{4}v^2\right] \text{ (Equivalent Radiators)} \quad (87)$$

If we define the effective internal radiation length t_r in the following way:

$$bt_r \equiv \frac{\alpha}{\pi}\left[\log\left(\frac{Q^2}{m^2}\right)-1\right] \quad (88)$$

then the effective internal bremsstrahlung spectum is given by:

$$\phi(v) \equiv \frac{[bt_r\phi(v)]_{\text{InB}}}{bt_r} \quad (89)$$

We'll follow Stein and TSAI71 and choose the equivalent radiators method as our "peaking" approximation:

$$\phi(v) = 1+v-\frac{3}{4}v^2 \quad (90)$$

The tail due to external bremstrahlung can be written in a similar form:

$$\left[\frac{d^2\sigma}{dE_p d\Omega}\right]_{\text{ExB}} = \left[\frac{d^2\sigma}{dE_p d\Omega}\right]^{\text{bef}}_{\text{ExB}} + \left[\frac{d^2\sigma}{dE_p d\Omega}\right]^{\text{aft}}_{\text{ExB}} \quad (91)$$

$$\left[\frac{d^2\sigma}{dE_p d\Omega}\right]^{\text{bef}}_{\text{ExB}} = \left[\frac{\eta_s-(\eta_s-1)\,v_s}{\eta_p}\right]\left[\frac{bt_b\phi(v_s)}{\omega_s}\right]\bar{F}(Q_s^2)\left[\frac{d\sigma(E_s-\omega_s)}{d\Omega}\right]_{\text{el}} \quad (92)$$

$$\left[\frac{d^2\sigma}{dE_p d\Omega}\right]^{\text{aft}}_{\text{ExB}} = \left[\frac{bt_a\phi(v_p)}{\omega_p}\right]\bar{F}(Q_p^2)\left[\frac{d\sigma(E_s)}{d\Omega}\right]_{\text{el}} \quad (93)$$

and the the tail due to collisional loss can be written in a similar form as well:

$$\left[\frac{d^2\sigma}{dE_p d\Omega}\right]_{\text{coll}} = \left[\frac{d^2\sigma}{dE_p d\Omega}\right]^{\text{bef}}_{\text{coll}} + \left[\frac{d^2\sigma}{dE_p d\Omega}\right]^{\text{aft}}_{\text{coll}} \quad (94)$$

$$\left[\frac{d^2\sigma}{dE_p d\Omega}\right]^{\text{bef}}_{\text{coll}} = \left[\frac{\eta_s-(\eta_s-1)\,v_s}{\eta_p}\right]\left[\frac{\xi_b}{\omega_s^2}\right]\bar{F}(Q_s^2)\left[\frac{d\sigma(E_s-\omega_s)}{d\Omega}\right]_{\text{el}} \quad (95)$$

$$\left[\frac{d^2\sigma}{dE_p d\Omega}\right]^{\text{aft}}_{\text{coll}} = \left[\frac{\xi_a}{\omega_p^2}\right]\bar{F}(Q_p^2)\left[\frac{d\sigma(E_s)}{d\Omega}\right]_{\text{el}} \quad (96)$$

where the collisional energy loss factor is given by

$$\xi_{b,a} = \sum_j^{b,a}\left[\frac{\xi}{\rho x}\right]_j \rho_j x_j = \sum_j^{b,a}\left(\sum_k\left[\frac{\xi}{\rho x}\right]_k w_k^j\right)\rho_j x_j \quad (97)$$

$$\left[\frac{\xi}{\rho x}\right]_k = \frac{Z_k a}{A_k\beta^2} \quad (98)$$

where w_k^j is the fraction by weight of the k-th element in the j-th material before or after scattering. Note that TSAI71 defines this factor as:

$$\xi_a = \xi_b \equiv \frac{\xi}{2} = \frac{\pi m}{4\alpha}\frac{t_a+t_b}{ZL+L'} \quad (99)$$

Specifically TSAI71 assumes the the effective collisional loss thickness is the same before and after scattering. In our case the thickness after scattering is always substantially larger ($t_a/t_b \geq 10 \gg 1$). TSAI71 also assumes the the collisional thickensss is directly proportional to the radiation thickness. If all of the target and window (non-target) material were composed of a single element, then this would be true. However, we have many different composite materials so this is not a reasonable assumption either.

The main differences and new corrections are:

1. the energy loss to colliions will be calculated using the most probable formula as opposed to the mean formula
2. we will use $b = 4/3$ and calculate the radiation lengths neglecting the extra small factor
3. the normalization factor will be written as a product of two gamma functions
4. the collisional loss factor will be calculate from scratch as opposed to assuming that it is proportional to the radiation length
5. for all the kinematic factors we'll use the mass of the nucleus as opposed to the mass of the atom
6. we'll include the factor for target radiation ala Miller 72
7. we'll keep the δ_0 as a sum of two Spence functions instead of using the approximate reduced form

6 Inelastic Tail

The results of the previous section allow us to subtract out the tail due to the elastic scattering from the target nucleus. What remains is the quasielastic and inelastic spectrum for scattering from the nucleon. Stein uses a model for the quasielastic and subtracts it our before coniderign the inelastic specrturm. We will treat the quasielastic region as if it were part of the inelastic spectrum. I believe that this is standard partice within the Pol 3He collaboration. As mentioned before, the area integral is reduced to two line integrals by the energy peaking approximation. This is tantamount to assuming that the the electron either radiates before or after scattering only. The error due to approximation scale as $b^2(t_a + t_b)^2$. Therefore the error is on order of a percent when $b(t_a + t_b) = 0.1$. The end result is a line integral over E'_p for fixed E_s, a line integral over E'_s over fixed E_p, and a small area integral over in the vicinity of E_s and E_p.

$$\left[\frac{d^2\sigma}{dEd\Omega}\right]_{\text{inelastic}} = \left[\frac{d^2\sigma}{dEd\Omega}\right]_{\text{aft}} + \left[\frac{d^2\sigma}{dEd\Omega}\right]_{\text{bef}} + \left[\frac{d^2\sigma}{dEd\Omega}\right]_{\text{low}} \tag{100}$$

The small area integral corresponds to the emission of very low energy photons. To evaluate this part, we assume that the cross section is constant over this bin, find the asymptotic forms of the straggling functions as the energy loss approaches zero, and then finally integrate over an energy bin of size ΔE by $R\Delta E$ where R is given by:

$$R = \frac{\eta_s}{\eta_p} = \frac{M_T + 2E_s \sin^2\left(\frac{\theta}{2}\right)}{M_T - 2E_p \sin^2\left(\frac{\theta}{2}\right)} \tag{101}$$

Accroding to Tsai 71, this bin size needs to be at least 10 times larger that the collisional energy loss factor $\Delta E/\xi \geq 10$ but small enough that the cross section does not change much over the bin. If done properly, the final answer should be insensitive to the choice of ΔE. The low photon energy integral is given by:

$$\left[\frac{d^2\sigma}{dEd\Omega}\right]_{\text{low}} = \left(\frac{R\Delta E}{E_s}\right)^{bt_b+bt_r} \left(\frac{\Delta E}{E_p}\right)^{bt_a+bt_r} \left[1 - \frac{(\xi_b + \xi_a)/\Delta E}{1 - b\,(t_a + t_b + 2t_r)}\right] \bar{F}(Q^2) \left[\frac{d^2\sigma}{dEd\Omega}\right]_{\text{born}} \tag{102}$$

The integral corresponding to energy loss only before scattering is given by:

$$\begin{aligned}
\left[\frac{d^2\sigma}{dEd\Omega}\right]_{\text{bef}} &= \int_{E_s^{\min}}^{E_s^{\max}} F^s_{\text{soft}}\,(W^s_b + W^s_i)\,\bar{F}(Q^2) \left[\frac{d^2\sigma}{dEd\Omega}\right]_{\text{born}} dE'_s && (103)\\
E_s^{\max} &= E_s - R\Delta E && (104)\\
E_s^{\min} &= E_p/\eta_p && (105)\\
F^s_{\text{soft}} &= \left(\frac{E_s - E'_s}{E_p R}\right)^{bt_a+bt_r} \left(\frac{E_s - E'_s}{E_s}\right)^{bt_b+bt_r} && (106)\\
W^s_b &= \left[\frac{bt_b + bt_r}{E_s - E'_s}\right] \phi\left(\frac{E_s - E'_s}{E_s}\right) && (107)\\
W^s_i &= \frac{\xi_b}{(E_s - E'_s)^2} && (108)
\end{aligned}$$

The integral corresponding to energy loss only after scattering is given by:

$$\begin{aligned}
\left[\frac{d^2\sigma}{dEd\Omega}\right]_{\text{aft}} &= \int_{E_p^{\min}}^{E_p^{\max}} F_{\text{soft}}^p \left(W_b^p + W_i^p\right) \bar{F}(Q^2) \left[\frac{d^2\sigma}{dEd\Omega}\right]_{\text{born}} dE_p' && (109)\\
E_p^{\max} &= E_s/\eta_s && (110)\\
E_p^{\min} &= E_p + \Delta E && (111)\\
F_{\text{soft}}^p &= \left(\frac{E_p' - E_p}{E_p'}\right)^{bt_a + bt_r} \left(\frac{(E_p' - E_p)R}{E_s}\right)^{bt_b + bt_r} && (112)\\
W_b^p &= \left[\frac{bt_a + bt_r}{E_p' - E_p}\right] \phi\left(\frac{E_p' - E_p}{E_p'}\right) && (113)\\
W_i^p &= \frac{\xi_a}{\left(E_p' - E_p\right)^2} && (114)
\end{aligned}$$

7 Spence Function

The Spence function is really a definite integral that must be evaluated numerically:

$$\Phi(x) = -\int_0^x \frac{\log|1-y|}{y}\, dy \tag{115}$$

AS 27.7 Dilogarithm lists the following properties:

$$\begin{aligned}
\Phi(x) &= \sum_{n=1}^{\infty} \frac{x^n}{n^2} \text{ for } |x| \leq 1 && (116)\\
\Phi(+1) &= \frac{\pi^2}{6} && (117)\\
\Phi(+1/2) &= \frac{\pi^2}{12} - \frac{1}{2}\log^2(2) && (118)\\
\Phi(0) &= 0 && (119)\\
\Phi(-1) &= -\frac{\pi^2}{12} && (120)\\
\Phi(x) &= +\frac{\pi^2}{3} - \frac{1}{2}\log^2|x| - \Phi(1/x) \text{ for } x \geq 1 && (121)\\
\Phi(x) &= -\frac{\pi^2}{6} - \frac{1}{2}\log^2|x| - \Phi(1/x) \text{ for } x \leq -1 && (122)\\
\Phi(x) &= +\frac{\pi^2}{6} - \log(1-x)\log(x) - \Phi(1-x) \text{ for } 0 < x < +1 && (123)
\end{aligned}$$

which can implemented recursively on the computer as:

$$\Phi(x) = \left\{ \begin{array}{ll}
-\pi^2/6 - 0.5\log^2|x| - \Phi(1/x) & x < -1.005\\
\Phi'(x) & -1.005 \leq x < -0.995\\
\sum_{n=1}^{N_{\max}} x^n/n^2 & -0.995 \leq x < 0\\
0 & x = 0\\
\sum_{n=1}^{50} x^n/n^2 & 0 < x < +0.5\\
+\pi^2/12 - 0.5\log^2(2) & x = +0.5\\
+\pi^2/6 - \log(1-x)\log(x) - \Phi(1-x) & +0.5 < x < +1\\
+\pi^2/6 & x = +1\\
+\pi^2/3 - 0.5\log^2|x| - \Phi(1/x) & x > +1
\end{array} \right\} \tag{124}$$

where the we treat the region between -1.005 and 0 in the following way. Between -0.995 and 0, we simply perform a brute force sum. However we limit the number of terms $N_{\max}$ by proscribing a desired precision

ϵ in the following way:

$$\epsilon = 10^{-14} \geq x^{N_{\max}}/N_{\max}^2 \tag{125}$$

Bewteen -1.005 and -0.995, we fit the DiLog function evaluated by Maple with 20 digit precision to a fourth order polynomial:

$$\begin{aligned}
\Phi'(x) &= -\frac{\pi^2}{12}\left[1 + \sum_{n=1}^{4} c_n\,(x+1)^n\right] && (126)\\
c_1 &= -0.8427659547\text{E}{-00} && (127)\\
c_2 &= -0.1174192121\text{E}{-00} && (128)\\
c_3 &= -0.2698522084\text{E}{-01} && (129)\\
c_4 &= -0.1077307053\text{E}{-01} && (130)
\end{aligned}$$

Note that the parameterization gets better as x approaches -1 from either direction. Therefore only near -1.005 and -0.995 does this parameterization yields 10 digit precision. Everywhere else one can expect around 14 digit precision.

8 Gamma Function

The gamma function is also a definite integral that must be evaluated numerically:

$$\Gamma(z) = \int_0^\infty t^{z-1}\exp(-t)\,dt \tag{131}$$

AS 6.1 has the following properties:

$$\begin{aligned}
\Gamma(n+1) &= n! \text{ for integer } n > 0 && (132)\\
\Gamma(1) &= 1 && (133)\\
\Gamma(1-z) &= \frac{\pi}{\sin(\pi z)\Gamma(z)} && (134)
\end{aligned}$$

The Lanczos approximation is given as:

$$\begin{aligned}
\Gamma(1+z) &= \sqrt{2\pi}\,(z+g+1/2)^{z+1/2}\exp(z+g+1/2)\,A_g(z) && (135)\\
A_g(z) &= \left[\frac{1}{2}\right]p_0(g) + \left[\frac{z}{z+1}\right]p_1(g) + \left[\frac{z(z-1)}{(z+1)(z+2)}\right]p_2(g) + \left[\frac{z(z-1)(z-2)}{(z+1)(z+2)(z+3)}\right]p_2(g) + \cdots && (136)
\end{aligned}$$

where $\Re z > 0$, g is a freely chosen constant subject to the constraint $\Re(z+g+1/2) > 0$, and the coeffecients $p_k(g)$ are functions of g. The sum can be truncated and rewritten in the form:

$$A_g(z) = c_0 + \sum_{k=1}^{N-1}\frac{c_k}{z+k} + \epsilon \tag{137}$$

where a judicous choice of g and N can results in arbitrarily small ϵ. Note that this approximation is valid for all complex numbers when used in conjunction with the reflection formula. The GNU Scientific Library uses $g = 7$ and $N = 9$ with the following coefficients:

$$\begin{aligned}
A_7(z) &= c_0 + \sum_{k=1}^{8}\frac{c_k}{z-1+k} && (138)\\
c_0 &= +0.9999999999998099\text{E}{+00} && (139)\\
c_1 &= +6.7652036812188510\text{E}{+02} && (140)
\end{aligned}$$

E_p	"exact" numerical integration M&T	**R**	Δ	peaking approximation M&T	**R**	Δ	"exact"/peaking M&T	**R**	Δ
18.4	15.85	15.85	+0.00	15.85	15.85	+0.00	+0.00	+0.00	+0.00
17.5	1.884	1.884	+0.00	1.862	1.862	+0.00	+1.17	+1.17	+0.00
16.5	1.246	1.246	+0.00	1.179	1.179	+0.00	+5.38	+5.38	+0.00
10.0	5.011	5.012	−0.02	3.835	3.836	−0.03	+23.47	+23.46	−0.01
5.0	42.70	42.71	−0.02	42.03	42.03	+0.00	+1.57	+1.59	+0.02
1.5	581.9	582.5	−0.10	676.5	676.6	−0.01	−16.26	−16.15	+0.09
GeV	nb/GeV/sr		%	nb/GeV/sr		%	%	%	%
4.8	17.26	17.26	+0.00	17.26	17.26	+0.00	+0.00	+0.00	+0.00
4.5	4.533	4.533	+0.00	4.526	4.531	−0.11	+0.15	+0.04	−0.11
4.0	2.250	2.250	+0.00	2.236	2.248	−0.54	+0.62	+0.09	−0.54
2.5	1.665	1.665	+0.00	1.615	1.686	−4.40	+3.00	−1.26	−4.40
1.0	5.690	5.699	−0.16	5.226	6.086	−16.46	+8.15	−6.79	−16.27
GeV	μb/GeV/sr		%	μb/GeV/sr		%	%	%	%
0.980	3.733	3.734	−0.03	3.733	3.734	−0.03	+0.00	+0.00	+0.00
0.900	0.6244	0.6246	−0.03	0.6239	0.6247	−0.13	+0.08	−0.02	−0.10
0.700	0.2275	0.2276	−0.04	0.2247	0.2279	−1.42	+1.23	−0.13	−1.38
0.500	0.1934	0.1937	−0.16	0.1846	0.1940	−5.09	+4.55	−0.15	−4.93
0.300	0.3048	0.3079	−1.02	0.3080	0.3081	−0.03	−1.05	−0.06	+0.97
0.200	0.5435	0.5611	−3.24	0.5612	0.5612	+0.00	−3.26	−0.02	+3.14
GeV	mb/GeV/sr		%	mb/GeV/sr		%	%	%	%

Table 1: Comparison of Radiative tails from elastic ep scattering with $E_s = 20, 5, 1$ GeV and $\theta = 5^o$. M&T refers to Table III. of [?], **R** refers to the fortran code **ROSETAIL.F** , and Δ is the relative difference between the two. The peaking approximation that is implemented in **ROSETAIL.F** correponds to the "Equivalent Radiators" column in M&T.

$$c_2 = -1.2591392167224028\text{E}{+}03 \tag{141}$$
$$c_3 = +7.7132342877765313\text{E}{+}02 \tag{142}$$
$$c_4 = -1.7661502916214059\text{E}{+}02 \tag{143}$$
$$c_5 = +1.2507343278686905\text{E}{+}01 \tag{144}$$
$$c_6 = -1.3857109526572012\text{E}{-}01 \tag{145}$$
$$c_7 = +9.9843695780195716\text{E}{-}06 \tag{146}$$
$$c_8 = +1.5056327351493116\text{E}{-}07 \tag{147}$$

where we have shifted z in order to evaluate $\Gamma(z)$. Using these coefficients in the following recursive computer implementation yields at least 13 digits for all real z:

$$\Gamma(z) = \left\{ \begin{array}{ll} \pi(1-z)/\sin[\pi(1-z)]/\Gamma(2-z) & z<1 \\ 1 & z=1 \\ \exp\left[0.5\log(2\pi) + (z-0.5)\log(z-0.5+g) - z + 0.5 - g + \log(A_7(z))\right] & z>1 \end{array} \right\} \tag{148}$$

9 comparison to Mo Tsai