Formulae for Inclusive Electron Scattering

Jaideep Singh
University of Virginia
Version 0.01

February 18, 2007

Abstract

A collection of formulae for kinematics and cross sections etc.

Contents

1 Units

The speed of light c is set equal to 1. To retrieve physical units:

$$\begin{aligned}
\text{energy } E &\to E &\to E/c \qquad (1)\\
\text{mass } m &\to mc^2 &\to mc \qquad (2)\\
4-\text{momentum } \ p &\to pc &\to p \qquad (3)\\
\text{momentum } \ \vec{p} &\to \vec{p}c &\to \vec{p} \qquad (4)\\
\text{invariant mass squared } W^2 &\to W^2c^4 &\to W^2c^2 \qquad (5)\\
4-\text{momentum transferred squared } \ Q^2 &\to Q^2c^2 &\to Q^2 \qquad (6)
\end{aligned}$$

2 Kinematics

We'll use the notation of MT69 [?]. All noninvarient quantities are evaluated in the lab frame of reference. In this frame, the target particle is initially at rest. The 4-momenta of the incoming and outgoing electron are s and p. The 4-momentum of the virtual photon is q. The 4-momentum of the target particle is $p_{\rm i}$ and the total 4-momentum of the producs is $p_{\rm f}$. They are depicted in Fig. (1) and are related to each other by the conservation of energy and momentum:

$$\begin{aligned}
s &= (E_s, \vec{p}_s) \qquad (7)\\
p &= (E_p, \vec{p}_p) \qquad (8)\\
q &= s - p = (E_s - E_p, \vec{p}_s - \vec{p}_p) = (\nu, \vec{q}\,) \qquad (9)\\
p_{\rm i} &= (M, 0) \qquad (10)\\
p_{\rm f} &= s + p_{\rm i} - p = p_{\rm i} + q = (M + \nu, \vec{q}\,) \qquad (11)
\end{aligned}$$

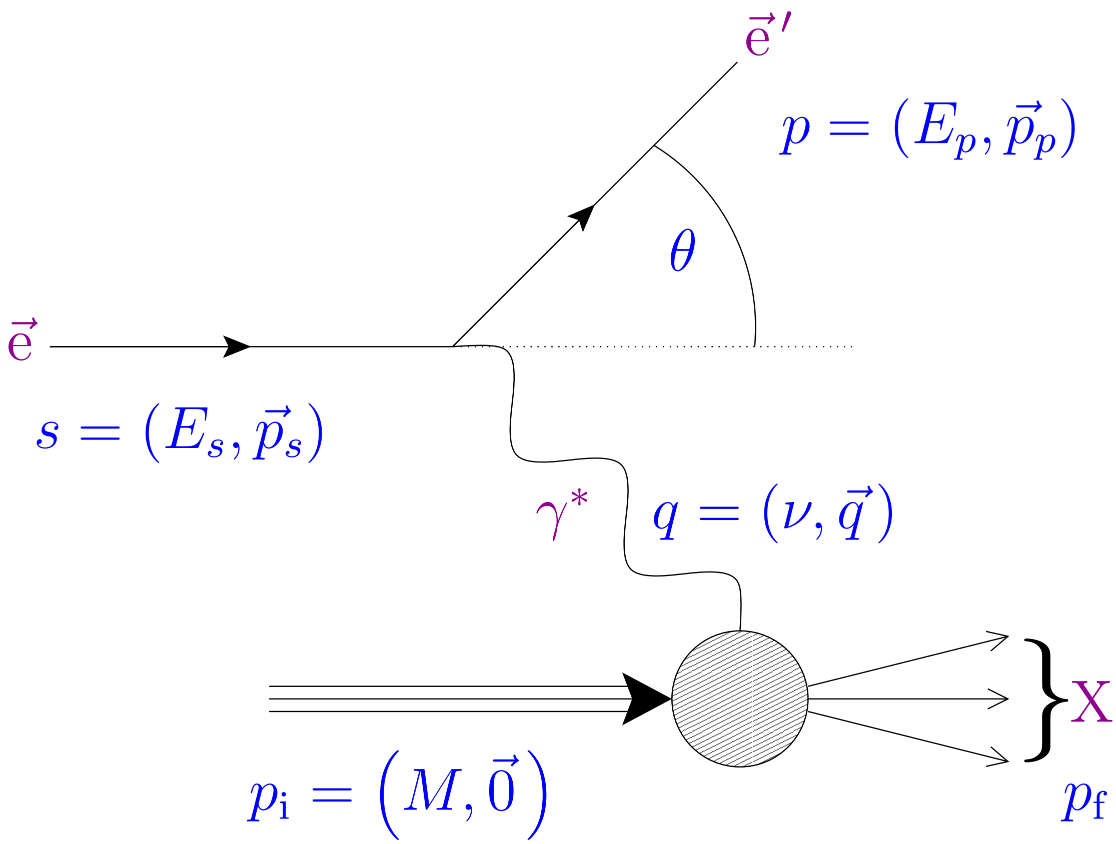

Figure 1: Kinematic Variables.

The invariant quantities are the rest masses of the electron m and target particle M, the imaginary rest mass of the virtual photon $i\sqrt{Q^2}$, and the sum of the rest masses of the products W:

$$
\begin{aligned}
s^2 &= E_s^2 - |\vec{p}_s|^2 = m^2 && (12)\\
p^2 &= E_p^2 - |\vec{p}_p|^2 = m^2 && (13)\\
q^2 &= \nu^2 - |\vec{q}|^2 = -Q^2 && (14)\\
p_{\mathrm{i}}^2 &= M^2 && (15)\\
p_{\mathrm{f}}^2 &= (M+\nu)^2 - |\vec{q}|^2 = W^2 && (16)
\end{aligned}
$$

The energy and momentum of the incoming and outgoing electron are:

$$
\begin{aligned}
p_s &= |\vec{p}_s|^2 && (17)\\
p_p &= |\vec{p}_p|^2 && (18)\\
E_s &= \sqrt{m^2 + p_s^2} && (19)\\
E_p &= \sqrt{m^2 + p_p^2} && (20)
\end{aligned}
$$

The energy lost by the incident electron is:

$$
\nu = E_s - E_p \qquad (21)
$$

The 4-momentum transferred squared is:

$$
Q^2 = -(s-p)^2 = -s^2 - p^2 + 2sp = -2m^2 + 2\left(E_s E_p - \vec{p}_s \cdot \vec{p}_p\right) \qquad (22)
$$

It is related to the angle of the scattered electron θ:

$$
\begin{aligned}
\vec{p}_s \cdot \vec{p}_p &= p_s p_p \cos(\theta) a && (23)\\
Q^2 &= 2\left[E_s E_p - p_s p_p \cos(\theta) - m^2\right] && (24)
\end{aligned}
$$

The invariant mass of the products is:

$$
\begin{aligned}
W^2 &= (M+\nu)^2 - |\vec{q}|^2 = \left(M^2 + 2\nu M + \nu^2\right) - \nu^2 - Q^2 && (25)\\
&= M^2 + 2\nu M - Q^2 && (26)
\end{aligned}
$$

It is useful to find a more direct relationship between ν and W. Inserting the equation for Q and isolating the p_p term gives:

$$
\frac{W^2 - M^2}{2} + \left(E_s E_p - m^2\right) - M\left(E_s - E_p\right) = p_s p_p \cos(\theta) \qquad (27)
$$

Squaring both sides results in a quadratic equation for E_p:

$$
\begin{aligned}
0 &= A E_p^2 - B E_p + C && (28)\\
A &= M^2 + m^2 + 2ME_s + p_s^2 \sin^2(\theta) && (29)\\
B &= \left[(M+E_s)\left(W^2 - M^2\right) + 2p_s^2\left(M + E_s \sin^2(\theta)\right)\right] && (30)\\
C &= \frac{\left(W^2 - M^2\right)^2}{4} + p_s^2\left(W^2 - M^2\right) + p_s^4 \sin^2(\theta) && (31)\\
E_p &= \frac{2C}{B + \sqrt{B^2 - 4AC}} && (32)
\end{aligned}
$$

Writing this in terms of ν gives:

$$
\nu = \frac{E_s\left(B + \sqrt{B^2 - 4AC}\right) - 2C}{B + \sqrt{B^2 - 4AC}} \qquad (33)
$$

3 Kinematic Limits

The minimum energy of the scattered electron is its mass:

$$E_p^{\min} = m \tag{34}$$

Elastic scattering occurs when target particle remains intact. The conservation of momentum requires that the electron loses some energy to the recoil of the target particle. Setting $W = M$ in equations (30) & (31) gives for B & C:

$$B_{\text{el}} = 2p_s^2 \left(M + E_s \sin^2(\theta)\right) \tag{35}$$

$$C_{\text{el}} = p_s^4 \sin^2(\theta) \tag{36}$$

Using these elastic values for B & C and rearranging some things gives for $E_p^{\max}$:

$$E_p^{\max} = E_s \left(\frac{1 + \cos(\theta)\sqrt{1 - \frac{m^2}{M^2}\sin^2(\theta)} + [1 + \cos(\theta)]\,\frac{2m^2}{ME_s}\sin^2\left(\frac{\theta}{2}\right)}{1 + \cos(\theta)\sqrt{1 - \frac{m^2}{M^2}\sin^2(\theta)} + [1 + \cos(\theta)]\,\frac{2E_s}{M}\sin^2\left(\frac{\theta}{2}\right)} \right) \tag{37}$$

Consequently, the minimum energy lost by the electron also depends on the scattering angle:

$$\nu_{\min} = E_s - E_p^{\max} \tag{38}$$

$$= E_s \left(\frac{[1 + \cos(\theta)]\,\frac{2p_s^2}{ME_s}\sin^2\left(\frac{\theta}{2}\right)}{1 + \cos(\theta)\sqrt{1 - \frac{m^2}{M^2}\sin^2(\theta)} + [1 + \cos(\theta)]\,\frac{2E_s}{M}\sin^2\left(\frac{\theta}{2}\right)} \right) \tag{39}$$

The maximum energy lost by the incident electron is:

$$\nu_{\max} = E_s - E_p^{\min} = E_s - m \tag{40}$$

The lowest Q^2 occurs when the electron loses most of it's energy:

$$Q^2_{\min} = 2m\nu_{\max} = 2m\left(E_s - m\right) \tag{41}$$

The highest Q^2 occurs when the electron is scattered elastically:

$$Q^2_{\max} = 2M\nu_{\min} \tag{42}$$

The lowest W also occurs for elastic scattering:

$$W^2_{\min} = M^2 + 2\nu_{\min}M - Q^2_{\max} = M^2 \tag{43}$$

The highest W occurs when the electron loses most of its energy:

$$W^2_{\max} = M^2 + 2\nu_{\max}M - Q^2_{\min} = M^2 + 2\left(E_s - m\right)\left(M - m\right) \tag{44}$$

4 Nonrelativistic Limit

The nonrelativistic limit is reached when $m \gg p$. The energy and momentum of the incoming and outgoing electron are:

$$E_s \simeq m + \frac{p_s^2}{2m} - \frac{p_s^4}{8m^3} \tag{45}$$

$$E_p \simeq m + \frac{p_p^2}{2m} - \frac{p_p^4}{8m^3} \tag{46}$$

The energy lost by the incident electron is:

$$\nu \simeq \frac{p_s^2 - p_p^2}{2m} + \frac{p_p^4 - p_s^4}{8m^3} \tag{47}$$

The 4-momentum transferred squared is:

$$Q^2 \simeq (\vec{p}_s - \vec{p}_p)^2 - 2\left(\frac{p_s^2 - p_p^2}{2m}\right)^2 \tag{48}$$

The invariant mass of the products is:

$$W^2 \simeq M^2 + \frac{M}{m}\left(p_s^2 - p_p^2 + \frac{p_p^4 - p_s^4}{4m^2}\right) - (\vec{p}_s - \vec{p}_p)^2 + 2\left(\frac{p_s^2 - p_p^2}{2m}\right)^2 \tag{49}$$

5 Relativistic Limit

The relativistic limit is reached when $m \ll p$. The energy and momentum of the incoming and outgoing electron are:

$$E_s \simeq p_s\left(1 + \frac{m^2}{2p_s^2}\right) \simeq p_s \tag{50}$$

$$E_p \simeq p_p\left(1 + \frac{m^2}{2p_p^2}\right) \simeq p_p \tag{51}$$

The energy lost by the incident electron is:

$$\nu \simeq (p_s - p_p)\left(1 - \frac{m^2}{2p_s p_p}\right) \simeq E_s - E_p \tag{52}$$

The 4-momentum transferred squared is:

$$Q^2 \simeq 4p_s p_p\left[\sin^2\left(\frac{\theta}{2}\right) + \left(\frac{m\,[p_s - p_p]}{2p_s p_p}\right)^2\right] \simeq 4E_s E_p \sin^2\left(\frac{\theta}{2}\right) \tag{53}$$

The invariant mass of the products is still:

$$W^2 = M^2 + 2\nu M - Q^2 \tag{54}$$

The minimum energy of the scattered electron is its mass:

$$E_p^{\min} = m \tag{55}$$

In this relativistic limit, the following relation holds for elastic scattering $W = M$:

$$W^2 = M^2 + 2\nu M - Q^2 \quad \rightarrow \quad Q^2 = 2M\nu \tag{56}$$

Rearranging some things gives for $E_p^{\max}$:

$$E_p^{\max} \simeq \frac{E_s}{1 + \frac{2E_s}{M}\sin^2\left(\frac{\theta}{2}\right)} \tag{57}$$

Consequently, the minimum energy lost by the electron also depends on the scattering angle:

$$\nu_{\min} = E_s - E_p^{\max} \simeq \frac{\frac{2E_s^2}{M}\sin^2\left(\frac{\theta}{2}\right)}{1 + \frac{2E_s}{M}\sin^2\left(\frac{\theta}{2}\right)} \tag{58}$$

The maximum energy lost by the incident electron is still:

$$\nu_{\max} \quad = \quad E_s - E_p^{\min} = E_s - m \tag{59}$$

The lowest Q^2 occurs when the electron loses most of it's energy and is still:

$$Q^2_{\min} \quad = \quad 2m\nu_{\max} = 2m\left(E_s - m\right) \tag{60}$$

The highest Q^2 occurs when the electron is scattered elastically:

$$Q^2_{\max} \quad = \quad 2M\nu_{\min} \simeq \frac{4E_s^2 \sin^2\left(\frac{\theta}{2}\right)}{1 + \frac{2E_s}{M}\sin^2\left(\frac{\theta}{2}\right)} \tag{61}$$

The lowest W occurs for elastic scattering and is still:

$$W^2_{\min} \quad = \quad M^2 + 2\nu_{\min}M - Q^2_{\max} = M^2 \tag{62}$$

The highest W occurs when the electron loses most of its energy:

$$W^2_{\max} \quad = \quad M^2 + 2\nu_{\max}M - Q^2_{\min} \simeq M^2\left(1 + \frac{2E_s}{M}\right) \tag{63}$$