Notes on the Geramisov-Drell-Hearn Sum Rule and it's Extensions to Finite Q^2

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Abstract

This note is a collection of stuff on the GDH sum rule and it's extenstions to finite Q^2. There are many conventions, units, and forms for doing and used in the integral. This note is meant to collect these things and stuff. Greatest abstract ever...

Contents

1 Magnetic Moments

1.1 Introduction

All units are in SI except where noted. To convert to DTU, set $\hbar = c = 1$. It is useful to measure atomic magnetic moments in units of the Bohr magneton:

$$\mu_{\mathrm{B}} = \tfrac{e\hbar}{2M_{\mathrm{e}}} \quad \text{(SI)} \qquad\qquad \mu_{\mathrm{B}} = \tfrac{e\hbar}{2M_{\mathrm{e}}c} \quad \text{(cgs)} \tag{1}$$

where e is the elementary unit of charge, M_{e} is the mass of the electron, $\hbar$ is Planck's constant divided by 2π, and c is the speed of light. One Bohr magneton is the magnetic moment expected for an electron at tree level, meaning the value before radiative corrections increase it by an amount on the order of α, the fine structure constant.

Nuclear magnetic moments are usually measured in units of nuclear magnetons:

$$\mu_{\mathrm{N}} = \tfrac{e\hbar}{2M_{\mathrm{p}}} \quad \text{(SI)} \qquad\qquad \mu_{\mathrm{N}} = \tfrac{e\hbar}{2M_{\mathrm{p}}c} \quad \text{(cgs)} \tag{2}$$

where M_{p} is the mass of the proton. One nuclear magneton is the magnetic moment expected for a proton at tree level if it were an ideal Dirac point particle. By "Dirac," I mean that it obeys the Dirac equation, by "point," I mean that it has no extended structure. This expectation is not true for the proton and this "anomaly" will be discussed later.

Electron spins (and the spins of other similar objects) are usually labelled by S. Nuclear spins are, by convention, labelled by I. Therefore, the nuclear magnetic moment operator is written as:

$$\dot{\vec{\mu}} = \mu \left(\frac{\dot{\vec{I}}}{I} \right) = \left(\frac{\mu}{I} \right) \dot{\vec{I}} \tag{3}$$

where I is the spin of the nucleus, and $\dot{\vec{I}}$ is the spin operator. For example, for $I = \frac{1}{2}$, the spin operator is composed of the Pauli matrices:

$$\dot{\vec{I}} = \frac{\dot{\vec{\sigma}}}{2} = \frac{1}{2} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \hat{\imath} + \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} \hat{\jmath} + \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \hat{k} \right] \tag{4}$$

and for $I = 1$, the spin operator is given as:

$$\dot{\vec{I}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \hat{\imath} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ +i & 0 & -i \\ 0 & +i & 0 \end{pmatrix} \hat{\jmath} + \begin{pmatrix} +1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \hat{k} \tag{5}$$

The *classical* magnetic moment for a charged particle in motion with charge Z, mass M, and angular momentum $\vec{L}$ is:

$$\vec{\mu} = \frac{Ze}{2M} \vec{L} \tag{6}$$

It is for this reason that the *intrinsic quantum mechanical* magnetic moment operator for a particle is written as:

$$\dot{\vec{\mu}} = g \frac{Ze\hbar}{2M} \dot{\vec{I}} \tag{7}$$

where g is called the Landé g-factor. The magnetic moment is simply:

$$\mu = g \frac{Ze\hbar}{2M} I \tag{8}$$

It is conventional to write g in units of nuclear (or Bohr) magnetons:

$$\frac{\mu}{I} = g\mu_{\rm N} \qquad \rightarrow \qquad \mu = g\mu_{\rm N} I \tag{9}$$

To avoid confusion, the two forms of g will be defined by:

$$\mu = g^{\star} \frac{Ze\hbar}{2M} I = g\mu_{\rm N} I \tag{10}$$

$$g^{\star} = \frac{g}{Z} \left(\frac{M}{M_{\rm P}} \right) \tag{11}$$

Note that for the electron & proton, $g = g^{\star}$, and for the neutron, $g^{\star}$ is undefined because an uncharged ideal point-like particle would not be expected to have a magnetic moment.

1.2 Ideal Point-Like Particle of Arbitrary Spin

For an ideal point-like particle of mass M, charge Z, and spin I, the expected magnetic moment (before radiative corrections) is:

$$\mu_{\rm point} = \frac{Ze\hbar}{M} I \quad \text{(SI)} \qquad \mu_{\rm point} = \frac{Ze\hbar}{Mc} I \quad \text{(cgs)} \qquad \mu_{\rm point} = 2Z \frac{M_{\rm P}}{M} \mu_{\rm N} I \tag{12}$$

which implies that $g^{\star} = 2$ for all ideal point-like particles regardless of spin. To be specific, by ideal point-like particle, I mean (following K.J.Kim and Y.S. Tsai):

1. The particle does not couple via the Strong force.

2. The particle obeys both P and T invariance, i.e., it does not have a permanent electric dipole moment.

3. We are considering the magnetic moment before radiative corrections are applied.

There is some disagreement in the literature about what the magnetic moment for such a particle should be.

In 1953, F.J. Belinfante [Phys. Rev. 92, 997 (1953)] conjectured that the magnetic moment for all ideal point-like particles is:

$$\mu_{\text{point}} = \frac{Ze\hbar}{2M} \tag{13}$$

which implies that $g^\star = \frac{1}{I}$. Multiple papers have been published such as the one by C.R. Hagen & W.J. Hurley [Phys. Rev. Lett. 24, 1381 (1970)] which contain proofs that $g^\star = \frac{1}{I}$. A key ingredient in proofs of that sort is the "minimal electromagnetic interaction" prescription:

1. Write down the free Lagrangian for a field.

2. Perform a local gauge transformation

3. Add terms that couple the field to the electromagnetic field and keep only the terms necessary to insure gauge invariance

W. Pauli showed that additonal terms could be added to the Lagrangian that were gauge invariant, but were not required to insure gauge invariance. [RMP 13 203 (1941)]

T.D. Lee showed that for fields with spins greater than $\frac{1}{2}$ that follow the "minimal electromagnetic interaction" prescription do not have unique couplings. For example, the magnetic dipole and electric quadrapole couplings for a spin 1 particle are not unique, but they do have a fixed relationship. [PR 140(4B) 967 (1965)]

It has been argued that "good" high energy behaviour of particles should fix their couplings. Specifically, S.Weinberg [1970 Brandeis Lecture] and K.J.Kim & Y.S. Tsai [Phys. Rev. D7, 3710 (1973)] have both made the argument that for an ideal point-like particle to have "good" high energy behaviour, it must obey the GDH sum rule, and therefore $g^\star$ must be equal to 2. H. Pagels [PR 158, 1566 (1967)] went a step further and argued that the forward Compton ampltude gives a *definition* of the charge and the anomalous magnetic moment of a particle:

> We also remark, as is well known, that the low-energy theorem provides one with an experimental definition of the total physical charge of a particle. For forward scattering it also provides an unambiguous definition of the anomalous magnetic moment of a particle and a sum rule for this quantity. Not only is this the case for spin $\frac{1}{2}$, but also evidently for higher-spin systems as well.

More recently the $g^\star = 2$ argument has been made by S. Ferrara, M. Porrati, & V.L. Telegedi [Phys. Rev. D46, 3529 (1992)] using string theory considerations and by S. Deser, A. Waldron, V. Pascalutsa [Phys. Rev. D62, 105031 (2000)] in their study of electrodynamics with massive particles with a spin of $\frac{3}{2}$.

In the Standard Model, additional ingredients (spontaneously broken gauge symmetries, renormalizabilty) add terms to the Lagrangian that are not strictly "minimal electromagnetic interactions," but that do fix $g^\star = 2$ for charged leptons and W bosons. To be absolutely clear, in the Standard Model, the magnetic moments of all the charged leptons and for the $W^\pm$ bosons are:

$$\mu_{\text{e}^\mp} = \mp\frac{e\hbar}{2M_{\text{e}}} \qquad \mu_{\mu^\mp} = \mp\frac{e\hbar}{2M_\mu} \qquad \mu_{\tau^\mp} = \mp\frac{e\hbar}{2M_\tau} \qquad \mu_{\text{W}^\pm} = \pm\frac{e\hbar}{M_{\text{W}}} \tag{14}$$

It is the magnetic moment of the $W^\pm$ that distinguishes between the two forms for the ideal point-like magnetic moment and establishes equation (12) as the corrent one. This has been verified experimentally (most famously for the electron) and the PDG2002 lists for the $W^\pm$ at tree level:

$$|g^\star_{W^\pm}| - 2 = \underbrace{(0.03 \pm 0.08)}_{\Delta\kappa_\gamma} + \underbrace{(-0.012 \pm 0.035)}_{\lambda_\gamma} \tag{15}$$

1.3 Anomalous Magnetic Moment

The anomalous magnetic moment is the part of the magnetic moment that differs from the expected ideal point-like magnetic moment. In other words, the anomalous magnetic moment should be called the anomalous *part* of the magnetic moment. The anomalous magnetic moment is usually written in such a way to indicate an excess of charge:

$$\mu = 2\left(Z\frac{M_{\mathrm{P}}}{M} + \kappa\right)\mu_{\mathrm{N}} I = (Z + \kappa^\star)\,\frac{e\hbar}{M} I \tag{16}$$

By convention it is a unitless quantity and is related to the g-factors in the following way:

$$\kappa^\star = Z\left(\frac{g^\star}{2} - 1\right) = \frac{g}{2}\left(\frac{M}{M_{\mathrm{P}}}\right) - Z \tag{17}$$

$$\kappa = Z\frac{M_{\mathrm{P}}}{M}\left(\frac{g^\star}{2} - 1\right) = \frac{g}{2} - Z\frac{M_{\mathrm{P}}}{M} \tag{18}$$

$$\frac{\kappa}{M_{\mathrm{P}}} = \frac{\kappa^\star}{M} \tag{19}$$

Recently X. Ji & Y. Li [Phys. Lett. B591, 76 (2004)] have suggested another definition of the anomalous magnetic moment:

$$\mu = (2ZI + \kappa_{\mathrm{Ji}})\,\frac{e\hbar}{2M} = \left[Z + \frac{\kappa_{\mathrm{Ji}}}{2I}\right]\frac{e\hbar}{M} I = \left(2Z + \frac{\kappa_{\mathrm{Ji}}}{I}\right)\frac{M_{\mathrm{P}}}{M}\mu_{\mathrm{N}} I \tag{20}$$

$$\kappa_{\mathrm{Ji}} = 2I\kappa^\star = 2I\left(\frac{M}{M_{\mathrm{P}}}\right)\kappa = gI\frac{M}{M_{\mathrm{P}}} - 2ZI \tag{21}$$

This definition is a generalization of the one made by H. Pagels [PR 158, 1566 (1967)] for a spin 1 particle with charge +1. Note that for the proton, all of the defintions become degenerate:

$$\kappa^{\mathrm{P}}_{\mathrm{Ji}} = \kappa^\star_{\mathrm{P}} = \kappa_{\mathrm{P}} = \frac{g_{\mathrm{P}}}{2} - 1 = +1.79285 \tag{22}$$

and for the neutron, they are all nearly degenerate because $M_{\mathrm{N}} \approx M_{\mathrm{P}}$:

$$\kappa^{\mathrm{n}}_{\mathrm{Ji}} = \kappa^\star_{\mathrm{n}} = \frac{g_{\mathrm{n}}}{2}\left(\frac{M_{\mathrm{n}}}{M_{\mathrm{P}}}\right) = -1.91568 \tag{23}$$

$$\approx \kappa_{\mathrm{n}} = \frac{g_{\mathrm{n}}}{2} = -1.91304 \tag{24}$$

Differences among these definitions truly manifest themselves when applied to nuclear particles with $Z > 1$, $M > M_{\mathrm{P}}$, and $I > \frac{1}{2}$. For example consider the ^{3}He nucleus:

$$\kappa^3_{\mathrm{Ji}} = \kappa^\star_3 = \frac{g_3}{2}\left(\frac{M_3}{M_{\mathrm{P}}}\right) - 2 = -8.36793 \tag{25}$$

$$\kappa_3 = \frac{g_3}{2} - \frac{2M_{\mathrm{P}}}{M_3} = -2.79569 \tag{26}$$

or for complete nondegeneracy among the definitions, consider the deuteron:

$$\kappa^d_{\mathrm{Ji}} = g_d\frac{M_d}{M_{\mathrm{P}}} - 2 = -0.285975 \tag{27}$$

$$\kappa^\star_d = \frac{g_d}{2}\left(\frac{M_d}{M_{\mathrm{P}}}\right) - 1 = -0.142987 \tag{28}$$

$$\kappa_d = \frac{g_d}{2} - \frac{M_{\mathrm{P}}}{M_d} = -0.0715291 \tag{29}$$

From an informal survey of the literature, unless otherwise noted, most authors are referring to $\kappa^\star$ when they refer to the anomalous magnetic moment. Note that for a spin 1 particle with charge +1, S.J. Brodsky & J.R. Hiller [PRD 46, 2141 (1992)] explicitly define the anomalous magnetic moment to be $\kappa^\star$.

2 Low Energy Theorems: Literature

2.1 Scattering of Light of Very Low Frequency by Systems of Spin $\frac{1}{2}$

[F.E. Low, PR 96, 1428 (1954)] The total real Compton scattering amplitude is [equation (1.1)]:

$$H' = \left(\frac{e^2}{m}\right)\mathbf{e}\cdot\mathbf{e}' - \left(\frac{ie}{m}\right)\left(2\mu - \frac{e}{m}\right)k\vec{\sigma}\cdot(\mathbf{e}'\times\mathbf{e}) \tag{30}$$

$$-2\frac{\mu^2}{k}i\vec{\sigma}\cdot\left[(\mathbf{e}\times\mathbf{k})\times\left(\mathbf{e}'\times\underbrace{\mathbf{k}}\right)\right] \tag{31}$$

$$-i\left(\frac{e}{m}\right)\frac{\mu}{k}\left[(\mathbf{e}\cdot\mathbf{k}')\,\mathbf{e}'\cdot(\vec{\sigma}\times\mathbf{k}') - (\mathbf{e}'\cdot\mathbf{k})\,\mathbf{e}\cdot(\vec{\sigma}\times\mathbf{k})\right] \tag{32}$$

Note that in the original text quoted above, there is a typo. Line (31) should have a prime on the last $\mathbf{k}$ giving the corrected equation:

$$H' = \left(\frac{e^2}{m}\right)\mathbf{e}\cdot\mathbf{e}' - \left(\frac{ie}{m}\right)\left(2\mu - \frac{e}{m}\right)k\vec{\sigma}\cdot(\mathbf{e}'\times\mathbf{e}) \tag{33}$$

$$-2\frac{\mu^2}{k}i\vec{\sigma}\cdot\left[(\mathbf{e}\times\mathbf{k})\times(\mathbf{e}'\times\mathbf{k}')\right] \tag{34}$$

$$-i\left(\frac{e}{m}\right)\frac{\mu}{k}\left[(\mathbf{e}\cdot\mathbf{k}')\,\mathbf{e}'\cdot(\vec{\sigma}\times\mathbf{k}') - (\mathbf{e}'\cdot\mathbf{k})\,\mathbf{e}\cdot(\vec{\sigma}\times\mathbf{k})\right] \tag{35}$$

For the forward ($\mathbf{k} = \mathbf{k}'$) real Compton scattering amplitude [4th line of equation (4.8)]:

$$H' = \left(\frac{e^2}{m}\right)\mathbf{e}\cdot\mathbf{e}' + i\omega\vec{\sigma}\cdot(\mathbf{e}'\times\mathbf{e})\,2\left(\mu - \frac{e}{2m}\right)^2 \tag{36}$$

2.2 Scattering of Low-Energy Photons by Particles of Spin $\frac{1}{2}$

[M. Gell-Mann & M.L. Goldberger, PR 96, 1433 (1954)] They derive the same relationship three different ways:

> (1) Classically, using Kramer's classical description of a Dirac particle; besides the Dirac magnetic moment considered by Kramers, a classical anomalous magnetic moment is added. (2) In quantum mechanics, for a Dirac particle with a Pauli anomalous magnetic moment, in the lowest order of pertubration theory. (3) *Exactly* in quantum field theory, for a Dirac particle interacting with arbitrary local and renormalizable fields, for example photons and mesons.
>
> The results of the three calculations are identical and show that the linear term depends only on the charge, mass, and static magnetic moment of the scattering particle.

The classical result [equation (2.15)]:

$$f = -\frac{e^2}{M}(\mathbf{e}'\cdot\mathbf{e}) - ig^2q\mathbf{S}\cdot[(\mathbf{n}'\times\mathbf{e}')\times(\mathbf{n}\times\mathbf{e})] \tag{37}$$

$$-\frac{ieg}{M}q\left[\mathbf{S}\cdot\left\{\frac{\mathbf{n}\,(\mathbf{n}\times\mathbf{e}) + (\mathbf{n}\times\mathbf{e})\,\mathbf{n}}{2}\right\}\cdot\mathbf{e}' - \mathbf{S}\cdot\left\{\frac{\mathbf{n}'\,(\mathbf{n}'\times\mathbf{e}') + (\mathbf{n}'\times\mathbf{e}')\,\mathbf{n}'}{2}\right\}\cdot\underbrace{\mathbf{e}'}\right] \tag{38}$$

$$+\frac{ieg_A}{M}q\mathbf{S}\cdot(\mathbf{e}'\times\mathbf{e}) \tag{39}$$

Note that in the original text quoted above, there is a typo. Line (38) should not have a prime on the last $\mathbf{e}'$ giving the corrected equation for the classical result:

$$f = -\frac{e^2}{M}(\mathbf{e}'\cdot\mathbf{e}) - ig^2q\mathbf{S}\cdot[(\mathbf{n}'\times\mathbf{e}')\times(\mathbf{n}\times\mathbf{e})] \tag{40}$$

$$-\frac{ieg}{M}q\left[\mathbf{S}\cdot\left\{\frac{\mathbf{n}\,(\mathbf{n}\times\mathbf{e}) + (\mathbf{n}\times\mathbf{e})\,\mathbf{n}}{2}\right\}\cdot\mathbf{e}' - \mathbf{S}\cdot\left\{\frac{\mathbf{n}'\,(\mathbf{n}'\times\mathbf{e}') + (\mathbf{n}'\times\mathbf{e}')\,\mathbf{n}'}{2}\right\}\cdot\mathbf{e}\right] \tag{41}$$

$$+\frac{ieg_A}{M}q\mathbf{S}\cdot(\mathbf{e}'\times\mathbf{e}) \tag{42}$$

The quantum mechanical result [equation (3.2)]:

$$f = -\frac{e^2}{M}(\mathbf{e}'\cdot\mathbf{e}) - 2i\mu^2 q\vec{\sigma}\cdot[(\mathbf{n}'\times\mathbf{e}')\times(\mathbf{n}\times\mathbf{e})] \tag{43}$$

$$-\frac{ie\mu}{M}q\left[\vec{\sigma}\cdot\left\{\frac{\mathbf{n}(\mathbf{n}\times\mathbf{e})+(\mathbf{n}\times\mathbf{e})\mathbf{n}}{2}\right\}\cdot\mathbf{e}' - \vec{\sigma}\cdot\left\{\frac{\mathbf{n}'(\mathbf{n}'\times\mathbf{e}')+(\mathbf{n}'\times\mathbf{e}')\mathbf{n}'}{2}\right\}\cdot\mathbf{e}\right] \tag{44}$$

$$+\frac{ie\mu_A}{M}q\vec{\sigma}\cdot(\mathbf{e}'\times\mathbf{e}) \tag{45}$$

which is in correspondence with the classical formula if the following identifications are made:

$$\mathbf{S}\leftrightarrow\frac{\vec{\sigma}}{2} \qquad g\leftrightarrow 2\mu \qquad g_A\leftrightarrow 2\mu_A \tag{46}$$

The quantum field theory result is a sum of $f^{(A)}$ [equation (3.14)] and $f^{(B)}$ [equation (3.16)]:

$$f^{(A)} = -\frac{e^2}{M}\mathbf{e}'\cdot\mathbf{e}F - 2i\frac{e^2}{4M^2}(F-F_2)^2 q_0\vec{\sigma}\cdot(\mathbf{n}'\times\mathbf{e}')\times(\mathbf{n}\times\mathbf{e}) \tag{47}$$

$$-\frac{ie}{M}\frac{e}{2M}(F-F_2)q_0[\vec{\sigma}\cdot\{\mathbf{nn}\times\mathbf{e}+\mathbf{n}\times\mathbf{en}\}\cdot\mathbf{e}' - \vec{\sigma}\cdot\{\mathbf{n}'\mathbf{n}'\times\mathbf{e}'+\mathbf{n}'\times\mathbf{e}'\mathbf{n}'\}\cdot\mathbf{e}] \tag{48}$$

$$+\frac{ie}{M}\frac{e}{2M}(F-F_2-1+4F_1)q_0\vec{\sigma}\cdot\mathbf{e}'\times\mathbf{e} \tag{49}$$

$$f^{(B)} = -\frac{e^2}{M}\mathbf{e}'\cdot\mathbf{e}(1-F) - \frac{ie}{M}\frac{eq_0}{2M}4F_1q_0\vec{\sigma}\cdot\mathbf{e}'\times\mathbf{e} \tag{50}$$

$$f = -\frac{e^2}{M}\mathbf{e}'\cdot\mathbf{e} - 2i\frac{e^2}{4M^2}(F-F_2)^2 q_0\vec{\sigma}\cdot(\mathbf{n}'\times\mathbf{e}')\times(\mathbf{n}\times\mathbf{e}) \tag{51}$$

$$-\frac{ie}{M}\frac{e}{2M}(F-F_2)q_0[\vec{\sigma}\cdot\{\mathbf{nn}\times\mathbf{e}+\mathbf{n}\times\mathbf{en}\}\cdot\mathbf{e}' - \vec{\sigma}\cdot\{\mathbf{n}'\mathbf{n}'\times\mathbf{e}'+\mathbf{n}'\times\mathbf{e}'\mathbf{n}'\}\cdot\mathbf{e}] \tag{52}$$

$$+\frac{ie}{M}\underbrace{\frac{e}{2M}(F-F_2-1)}_{\mu_A}q_0\vec{\sigma}\cdot\mathbf{e}'\times\mathbf{e} \tag{53}$$

2.3 Low-Energy Theorems for Spin $S \geq 1$

[A. Pais, PRL 19, 544 (1967)] He gives the Compton amplitude $A(\omega,\theta,\phi)$ as a multipole expansion defined by [equation (7)]:

$$A = \epsilon'_m A_{mn}\epsilon_n = \frac{1}{M}\sum_{NLM}\left(\frac{\omega}{M}\right)^N \epsilon'_m A_{mn}^{(NLM)}\epsilon_n Y_{LM}(\theta,\phi) \tag{54}$$

which to zeroth order in ω [equation (8)] and to first order in ω [equation (9)] gives:

$$A_{mn}^{(000)} = -e^2\delta_{mn}\sqrt{4\pi} \quad (\text{“Thomson”}) \tag{55}$$

$$A_{mn}^{(100)} = -\frac{1}{4}ie^2(g-2)^2\epsilon_{mnl}S_l\sqrt{4\pi} \tag{56}$$

2.4 Low-Energy Compton Scattering by Arbitrary-Spin Targets

I.J. Kalet [PR 176, 2135 (1968)] generalizes the Compton scattering amplitude [equation (58)]:

$$T = \frac{e^2}{m}\vec{\varepsilon'}\cdot\vec{\varepsilon} - 2i\omega\frac{e}{m}\left(\mu-\frac{e}{2m}\right)\vec{\varepsilon'}\times\vec{\varepsilon}\cdot\mathbf{S} \tag{57}$$

$$+\frac{ie\mu}{m\omega}\left[\left(\vec{\varepsilon'}\cdot\mathbf{k}\right)(\vec{\varepsilon}\cdot\mathbf{k}\times\mathbf{S}) - (\vec{\varepsilon}\cdot\mathbf{k}')\left(\vec{\varepsilon'}\cdot\mathbf{k}'\times\mathbf{S}\right)\right] \tag{58}$$

$$+\frac{i\mu^2}{\omega}\mathbf{S}\cdot\left[(\vec{\varepsilon}\times\mathbf{k})\times\left(\vec{\varepsilon'}\times\mathbf{k}'\right)\right] \tag{59}$$

2.5 Low-Energy Theorem for Compton Scattering

S. Saito [PR 184, 1894 (1969)] generalizes the Compton scattering amplitude [equation (5.15)]:

$$\sum_{\mu,\nu=1}^{4} \epsilon_\mu(\mathbf{k}',\beta')\ \epsilon_\nu(\mathbf{k},\beta)\, M_{m'\mu;m\nu}(\mathbf{p},\mathbf{Q}) = i\frac{e^2}{M}\left(\vec{\varepsilon'}\cdot\vec{\varepsilon}\right)\delta_{m'm} + \frac{e}{M}\left(2\mu - s\frac{e}{M}\right)k\left(\vec{\varepsilon'}\times\vec{\varepsilon}\right)\cdot\mathbf{S}_{m'm} \tag{60}$$

$$-\frac{e\mu}{kM}\left[(\vec{\varepsilon}\cdot\mathbf{k}')\left(\vec{\varepsilon'}\times\mathbf{k}'\right) - \left(\vec{\varepsilon'}\cdot\mathbf{k}\right)(\vec{\varepsilon}\times\mathbf{k})\right]\cdot\mathbf{S}_{m'm} \tag{61}$$

$$-\frac{\mu^2}{ks}\left[\left(\vec{\varepsilon'}\times\mathbf{k}'\right)\times\underbrace{(\vec{\varepsilon}+\mathbf{k})}\right]\mathbf{S}_{m'm} + O(k^2) \tag{62}$$

Note that there is a typo in line (62). The sum between $\mathbf{k}'$ and $\mathbf{S}_{m'm}$ should be a cross product:

$$\sum_{\mu,\nu=1}^{4} \epsilon_\mu(\mathbf{k}',\beta')\ \epsilon_\nu(\mathbf{k},\beta)\, M_{m'\mu;m\nu}(\mathbf{p},\mathbf{Q}) = i\frac{e^2}{M}\left(\vec{\varepsilon'}\cdot\vec{\varepsilon}\right)\delta_{m'm} + \frac{e}{M}\left(2\mu - s\frac{e}{M}\right)k\left(\vec{\varepsilon'}\times\vec{\varepsilon}\right)\cdot\mathbf{S}_{m'm} \tag{63}$$

$$-\frac{e\mu}{kM}\left[(\vec{\varepsilon}\cdot\mathbf{k}')\left(\vec{\varepsilon'}\times\mathbf{k}'\right) - \left(\vec{\varepsilon'}\cdot\mathbf{k}\right)(\vec{\varepsilon}\times\mathbf{k})\right]\cdot\mathbf{S}_{m'm} \tag{64}$$

$$-\frac{\mu^2}{ks}\left[\left(\vec{\varepsilon'}\times\mathbf{k}'\right)\times(\vec{\varepsilon}\times\mathbf{k})\right]\cdot\mathbf{S}_{m'm} + O(k^2) \tag{65}$$

where:

$$(S_{m'm})_\tau \equiv \left[\frac{(s+1)(2s+1)}{s}\right]^{\frac{1}{2}}(-1)^{s-m'}\underbrace{\begin{pmatrix} s & 1 & s \\ -m' & \tau & m \end{pmatrix}}_{\text{Wigner } 3j \text{ sybmol}} \tag{66}$$

3 The GDH Sum Rule: Original Papers

Here i write the equations as they appear exactly

3.1 Structure of the Proton and the Hyperfine Shift in Hydrogen

C.K. Iddings [PR 138, B446 (1965)] in the course of studing the hyperfine structure of Hydrogen may have derived the sum rule [equation (2.18)]. He does reference the Low and Gell-Mann & Goldberger papers regarding the Low Energy Theorem. It appears he had all the pieces (dispersion relations, virtual Compton scattering amplitude) but did not put them together explicitly. He thanks Drell and Hearn in his Acknowledgements and he was at SLAC at the same time as them. They must have known about his work and vice versa. The point is that his paper was recieved on October 26, 1964 and it was published April 26, 1965 which is before Gerasimov. This paper is referenced in D.A. Dicus & R. Vega [Phys. Lett. B 501, 44 (2001)]. In that paper they verify the GDH sum rule on the electron upto third order in α.

3.2 A Sum Rule for Magnetic Moments and the Damping of the Nucleon Magnetic Moment in Nuclei

The earliest known publication of the GDH sum rule is attributed to S.B. Gerasimov. It was submitted to *Yadernaya Fizika* on March 9, 1965 and published in the October 1965 issue [Yad. Fiz. 2, 598 (1965)]. The English translation by J.G. Adashko was published in the *Soviet Journal of Nuclear Physics* in April 1966 [Sov. J. Nucl. Phys. 2, 430 (1966)]. For a spin $\frac{1}{2}$ particle, the sum rule reads [equation (7)]:

$$\frac{2\pi^2 e^2}{M^2}g^2 = \int_{\omega_{\text{thr}}} \frac{\sigma_R(\omega) - \sigma_L(\omega)}{\omega}d\omega \tag{67}$$

where $\sigma_{R(L)}$ is the total cross section for the photon spin parallel (antiparallel) with the target spin and g is the anomalous magnetic moment of the nucleon in nuclear magnetons. Given this definition of g, M then has to be the mass of the proton.

Later in the paper, the sum rule is generalized for arbitrary spin [equation (15)]:

$$4\pi^2 S\left(\frac{1}{S}\mu_0 - \frac{Q}{M}\right)^2 = \int_{\omega_{\text{thr}}}^{\infty} \frac{\sigma_R(\omega) - \sigma_L(\omega)}{2} d\omega \tag{68}$$

3.3 Exact Sum Rule for Nucleon Magnetic Moments

The S.D. Drell and A.C. Hearn [PRL 16, 908 (1966)] paper was recieved April 20, 1966 and was published May 16, 1966 with the following form of the sum rule:

$$\int_0^{\infty} \frac{d\nu}{\nu}\left[\sigma_{\mathbf{P}}(\nu) - \sigma_{\mathbf{A}}(\nu)\right] = +\frac{2\pi^2\alpha}{M_p^2}\kappa_p^2 = 205\mu b \tag{69}$$

It is evaluated for the proton.

3.4 The Japanese Guys

M. Hosoda and K. Yamamoto in 1966 got as quoted in H. Pagels [PR 158, 1566 (1967)]:

$$4\pi^2\alpha\left\langle s_3\right\rangle\left(\frac{\mu}{s} - \frac{Z}{m}\right)^2 = \int_0^{\infty} \frac{dp}{p}\left[\sigma^+(p) - \sigma^-(p)\right] \tag{70}$$

4 The GDH Sum Rule for Spin $> \frac{1}{2}$: Literature

4.1 Decays $\pi^0 \to 2\gamma$,$\eta \to 2\gamma$ and Sum Rules for Nucleon Compton Scattering

H. Pagels [PR 158, 1566 (1967)] derives a the forward scattering Compton amplitude [equation (B4)] and GDH sum rule [equation (B5)] for a particle of spin 1, charge +1, and a magnetic moment of $\mu = (1+\kappa)\frac{e}{2m}$:

$$\lim_{p\to 0} 4\pi f = -\frac{e^2}{m}\vec{\varepsilon}_f^{\,*}\cdot\vec{\varepsilon}_i\;\vec{\lambda}_f^{*}\cdot\vec{\lambda}_i - \frac{e^2 p}{4m^2}(1-\kappa)^2\left(\vec{\varepsilon}_f^{\,*}\times\vec{\varepsilon}_i\right)\cdot\left(\vec{\lambda}_f^{*}\times\vec{\lambda}_i\right) + O\left(p^2\right) \tag{71}$$

$$\frac{\alpha\pi^2(1-\kappa)^2}{m^2} = \int_0^{\infty}\frac{dp}{p}\left[\sigma_+(p) - \sigma_-(p)\right] \tag{72}$$

4.2 Dynamic and Algebraic Symmetries

S. Weinberg [Lectures on Elementary Particles and Quantum Field Theory, 1970 Brandeis University Summer Institute in Theortical Physics, Volume 1, edited by S. Deser, M. Grisaru, & H. Pendleton, MIT Press, Cambridge, Mass. (1970), p285-393] derives the Compton scattering amplitude [equation (2.I.19)]:

$$f\left(\mathbf{q}'\lambda', \mathbf{q}\lambda\right) \to -\frac{e^2}{4\pi m}\mathbf{e}'^*\cdot\mathbf{e}1 - \frac{ieq^0}{2\pi m}\left(\frac{\mu}{J} - \frac{e}{2m}\right)\left(\mathbf{e}'^*\times\mathbf{e}\right)\cdot\mathbf{J} \tag{73}$$

$$-i\left(\frac{e}{m}\right)\frac{\mu}{k}\left[\left(\mathbf{e}\cdot\mathbf{k}'\right)\mathbf{e}'\cdot\left(\vec{\sigma}\times\mathbf{k}'\right) - \left(\mathbf{e}'\cdot\mathbf{k}\right)\mathbf{e}\cdot\left(\vec{\sigma}\times\mathbf{k}\right)\right] \tag{74}$$

$$-\frac{\mu^2}{4\pi J^2 q^0}\left[\mathbf{e}'^*\cdot\left(\mathbf{J}\times\mathbf{q}'\right)\underbrace{,}\;\mathbf{e}\cdot\left(\mathbf{J}\times\mathbf{q}\right)\right] \tag{75}$$

Note that there is a typo on line (75) labeled by the underbrace. It should be a cross product:

$$f\left(\mathbf{q}'\lambda', \mathbf{q}\lambda\right) \to -\frac{e^2}{4\pi m}\mathbf{e}'^*\cdot\mathbf{e}1 - \frac{ieq^0}{2\pi m}\left(\frac{\mu}{J} - \frac{e}{2m}\right)\left(\mathbf{e}'^*\times\mathbf{e}\right)\cdot\mathbf{J} \tag{76}$$

$$-i\left(\frac{e}{m}\right)\frac{\mu}{k}\left[\left(\mathbf{e}\cdot\mathbf{k}'\right)\mathbf{e}'\cdot\left(\vec{\sigma}\times\mathbf{k}'\right) - \left(\mathbf{e}'\cdot\mathbf{k}\right)\mathbf{e}\cdot\left(\vec{\sigma}\times\mathbf{k}\right)\right] \tag{77}$$

$$-\frac{\mu^2}{4\pi J^2 q^0}\left[\mathbf{e}'^*\cdot\left(\mathbf{J}\times\mathbf{q}'\right)\times\mathbf{e}\cdot\left(\mathbf{J}\times\mathbf{q}\right)\right] \tag{78}$$

Noting the following identities for scattering in the forward direction:

$$\mathbf{q} = \mathbf{q}' = (0, 0, \omega) \tag{79}$$
$$q^0 \equiv \omega \tag{80}$$
$$\lambda = \lambda' = \pm 1 \tag{81}$$
$$\mathbf{e} = \mathbf{e}' = \frac{1}{\sqrt{2}} (1, i\lambda, 0) \tag{82}$$

the forward scattering amplitude becomes [equation(2.I.20)]:

$$f(\omega, \lambda) \to -\frac{e^2}{4\pi m} 1 + \frac{\lambda J_z \omega}{4\pi} \left(\frac{\mu}{J} - \frac{e}{m} \right)^2 + O\left(\omega^2\right) \tag{83}$$

There may be some missing factors of i in the above equations. Regardless, the GDH sum rule is [equation (2.I.15)]:

$$\pi J_z \left(\frac{\mu}{J} - \frac{e}{m} \right)^2 = \int_0^\infty \frac{[\sigma(\omega, +1) - \sigma(\omega, -1)]}{\omega} \mathrm{d}\omega \tag{84}$$

4.3 Electromagnetic Interactions of Weakly-Bound Composite Systems

F.E. Close & L.A. Copley [Nucl. Phys. B19, 477 (1970)] derive the forward Compton amplitude [equation (43)]:

$$-f(\omega) = e^2 \frac{\vec{\varepsilon'}^* \cdot \vec{\varepsilon}}{M'} + i\omega \left(\frac{\mu}{S} - \frac{e}{M'} \right)^2 \mathbf{S} \cdot \vec{\varepsilon'}^* \times \vec{\varepsilon} \tag{85}$$

The GDH sum rule is [equation (44)]:

$$J_{N'} \equiv \int_0^\infty \frac{\mathrm{d}\omega}{\omega} \left[\sigma^P_{N'}(\omega) - \sigma^A_{N'}(\omega) \right] = \pi \left\langle \mathbf{S} \cdot \hat{\mathbf{q}} \right\rangle_{N'} \left(\frac{\mu_{N'}}{S} - \frac{Ze}{M'} \right)^2 \tag{86}$$

4.4 Low-energy theorem for Compton Scattering and the Drell-Hearn-Gerasimov sum rule: Exchange Currents

J.L. Friar [PRC 16, 1504 (1977)] derived the following Compton scattering amplitude for a nucleus [equation (29)]:

$$T = \frac{Z^2 \vec{\epsilon} \cdot \vec{\epsilon'}}{M_B} + i\omega\bar{\mu}^2 \vec{S} \cdot \left(\vec{\epsilon'} \times \hat{k}' \right) \times \left(\vec{\epsilon} \times \hat{k} \right) \tag{87}$$
$$-i \frac{Z\bar{\mu}\omega}{M_t} \left[\left(\vec{\epsilon'} \cdot \hat{k} \right) \vec{S} \cdot \vec{\epsilon} \times \hat{k} - \vec{\epsilon} \cdot \hat{k}' \left(\vec{S} \cdot \vec{\epsilon'} \times \hat{k}' \right) \right] \tag{88}$$
$$+ \frac{2i\omega Z}{M_t} \left(\bar{\mu} - \frac{Z}{2M_t} \right) \vec{S} \cdot \vec{\epsilon} \times \vec{\epsilon'} \tag{89}$$

where M_B is the total nuclear mass, M_t is the nuclear mass excluding the binding energy, and $\bar{\mu} \equiv \frac{\mu}{S}$ [equation (21)]. Using the approximation that $M_t \approx M_B$ and considering forward scattering yields the forward scattering amplitude and the GDH sum rule respectively [equations (1),(2a),(2b),& (4)]:

$$f(\omega) = \vec{\epsilon} \cdot \vec{\epsilon'} f_1(\omega) + i\omega f_2(\omega) \vec{S} \cdot \vec{\epsilon'} \times \vec{\epsilon} \tag{90}$$
$$f_1(0) = \frac{Z^2}{M_B} \tag{91}$$
$$f_2(0) = \left(\frac{\mu}{S} - \frac{Z}{M_B} \right)^2 \tag{92}$$
$$\int_{\omega_{th}}^\infty (\sigma_P - \sigma_A) \frac{d\omega'}{\omega'} = 4\pi^2 \alpha S \left(\frac{\mu}{S} - \frac{Z}{M_B} \right)^2 \tag{93}$$

4.5 Universal properties of the electromagnetic interactions of spin-one systems

S.J. Brodsky & J.R. Hiller [PRD 46, 2141 (1992)] explicitly define the anomalous magnetic moment of a spin one system to be $\mu_a = \mu_1 - \frac{e}{M}$, which leads to the corresponding GDH Sum Rule:

$$\mu_a^2 \quad = \quad \frac{1}{\pi}\int_{\omega_{th}}^{\infty} \frac{d\omega}{\omega}\left[\sigma_P(\omega) - \sigma_A(\omega)\right] \tag{94}$$

4.6 Sum rules and spin-dependent polarizabilities of the deuteron in effective field theory

X. Ji & Y. Li [Phys. Lett. B 591, 76 (2004)] consider the GDH sum rule for an arbitrary spin target summed over all magnetic states. As discussed before, they define the anomalous magnetic moment as $\kappa = \mu - 2S$ in units of $\frac{e\hbar}{2Mc}$. Defining the GDH integrand as:

$$\sigma_1 = \left[\frac{3}{S(S+1)(2S+1)}\right]\sum_{m_S} m_S \sigma_{m_S} \tag{95}$$

which gives the following sum rule [equation (27)]:

$$\frac{\alpha_{\rm em}\kappa^2}{4S^2M^2} = \frac{1}{2\pi^2}\int_0^{\infty} d\omega' \frac{\sigma_1(\omega')}{\omega'} \tag{96}$$

5 The GDH Sum Rule for Bound Systems: Literature

Brodksy argued with a bunch of people that the sum rule should be valid for a bound composite systems in addition to nucleons. As far as i'm, concerned he won and it is valid for composite/bound particles. Gerasimov in his original paper says without proof that the sum rule is true for nuclei as well and in fact uses the sum rule to study binding effects.

N. Dombey [PRL 19, 985 (1967)]
G. Barton & N. Dombey [PR 162, 1520 (1967)]
E.A. Peterson [PRL 20, 776 (1968)]
S.J. Brodsky & J.R. Primack [PR 174, 2071 (1968)]
R.A. Krajcik & L.L. Foldy [PRL 24, 545 (1970)]
F.E. Close & L.A. Copley [Nucl. Phys. B19, 477 (1970)]

6 Derivation

6.1 Low Energy Theorem

derive LET

6.2 Forward Scattering Compton Amplitude

For forward scattering the incoming and outgoing 4-momenta are set equal $q' = q$ where $q = (q_0, \vec{q})$ and $\nu \equiv q_0$, which yields the real photon forward Compton amplitude to first order in ν:

$$i\mathcal{M}(\nu) = -i\frac{Z^2e^2}{2M}\left(\vec{\varepsilon'}^* \cdot \vec{\varepsilon}\right) + \frac{\nu}{2}\left(\frac{\mu}{I} - \frac{Ze}{M}\right)^2 \vec{I}\cdot\left(\vec{\varepsilon'}^* \times \vec{\varepsilon}\right) + O(\nu^2) \tag{97}$$

By convention, the photon propagation axis will be taken as the z-axis, $\hat{q} = \hat{z} = \hat{k}$. The classical electric field polarization vector for a circularly polarized real photon is given by:

$$\vec{\varepsilon}_\pm = \vec{\varepsilon}^*_\mp = \frac{1}{\sqrt{2}}(\hat{x} \pm i\hat{y}) = \frac{1}{\sqrt{2}}(1, \pm i, 0) \tag{98}$$

where $+(-)$ is for right (left) circularly polarized light. Quantum mechanically the two states of circular polarization of the light can be associated with the two eigenstates of the photon spin operator $\dot{\vec{S}}_{\text{photon}} = \dot{\vec{\sigma}}$:

$$\dot{\vec{S}}^2{}_{\text{photon}} \, |\varepsilon_\pm\rangle \quad = \quad 2 \, |\varepsilon_\pm\rangle \tag{99}$$

$$\dot{S}_{z\,\text{photon}} \, |\varepsilon_\pm\rangle \quad = \quad \pm \, |\varepsilon_\pm\rangle \tag{100}$$

For forward scattering $\vec{\varepsilon'} = \vec{\varepsilon}$, therefore the dot and cross products of the initial and final polarization vectors are:

$$\vec{\varepsilon'}^*_\pm \cdot \vec{\varepsilon}_\pm = \frac{1}{\sqrt{2}}(1, \mp i, 0) \cdot \frac{1}{\sqrt{2}}(1, \pm i, 0) = \frac{1 - i^2}{2} = 1 \tag{101}$$

$$\vec{\varepsilon'}^*_\pm \times \vec{\varepsilon}_\pm = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{1}{\sqrt{2}} & \frac{\mp i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{\pm i}{\sqrt{2}} & 0 \end{vmatrix} = +\frac{\hat{z}}{2} \begin{vmatrix} 1 & \mp i \\ 1 & \pm i \end{vmatrix} = \pm i \hat{z} \tag{102}$$

Given these expressions, the helicity of a photon can be defined both classically and quantum mechanically:

$$h = (\vec{\varepsilon}^* \times \vec{\varepsilon}) \cdot \hat{q} \qquad \text{classical} \tag{103}$$

$$h = \langle\varepsilon| \, \dot{\vec{S}}_{\text{photon}} \cdot \hat{q} \, |\varepsilon\rangle \qquad \text{QM} \tag{104}$$

Note that the QM expression for helicity is true for *any* particle with spin operator $\dot{\vec{S}}$ and propagation direction $\hat{q}$. For a real photon ($\hat{q} \cdot \vec{\varepsilon} = 0$) that is in a helicity eigenstate, the helicity is ± 1. To be absolutely clear, a right (left) circularly polarized photon has its spin aligned parallel (antiparallel) to it's propagation direction and therefore has a helicity of $+1(-1)$. Putting these values in gives the photon helicity dependant forward Compton amplitude:

$$i\mathcal{M}^\pm(\nu) = -i\frac{Z^2e^2}{2M} \pm i\frac{\nu}{2}\left(\frac{\mu}{I} - \frac{Ze}{M}\right)^2 \vec{I} \cdot \hat{z} + O(\nu^2) \tag{105}$$

To evaluate the projection of the target spin $\vec{I}$ along the photon propagation axis $\hat{q} = \hat{z}$, we'll replace $\vec{I} \cdot \hat{z}$ with it's corresponding QM expectation value $\left\langle \dot{\vec{I}} \cdot \hat{q} \right\rangle$ where the target spin state is described by $|t\rangle$:

$$\dot{\vec{I}}^2 \, |t\rangle = I(I+1) \, |t\rangle \tag{106}$$

$$\dot{I}_z \, |t\rangle = m_I |t\rangle \, , \quad m_I = -I \ldots I \tag{107}$$

$$\left\langle \dot{\vec{I}} \cdot \hat{q} \right\rangle = \langle t| \, \dot{\vec{I}} \cdot \hat{z} \, |t\rangle = \langle t| \, \dot{I}_z \, |t\rangle = m_I \tag{108}$$

Putting this all together and labelling the forward Compton amplitude by order in ν and by *photon helicity*:

$$\mathcal{M}^\pm = \mathcal{M}_0^\pm + \nu\mathcal{M}_1^\pm + O(\nu^2) \tag{109}$$

$$\mathcal{M}_0^\pm = -\frac{Z^2e^2}{2M} \tag{110}$$

$$\mathcal{M}_1^\pm = \pm\frac{m_I}{2}\left(\frac{\mu}{I} - \frac{Ze}{M}\right)^2 \tag{111}$$

6.3 The Optical Theorem

do the things that shows that an imaginary index of refraction gives an exponential decaying factor to amplitude of the input wave The Optical Theorem relates the imaginary part of the scattering amplitude with the total scattering cross section:

$$\text{Im}\,\mathcal{M}^\pm(\nu) = \frac{\nu}{2}\sigma^\pm(\nu) \tag{112}$$

6.4 Causality, Analycity, & Dispersion Relations

the regular stuff

$$\mathrm{Re}\ f(z) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\mathrm{Im}\ f(z}{} \tag{113}$$

6.5 Crossing Symmetry

i think this is what lets you reduce the integral $\int_{-\infty}^{\infty} \to 2\int_0^{\infty}$

6.6 The Sum Rule

Putting this all together we get in (D.T.U.):

$$I_{\mathrm{GDH}} = \int_0^{\infty} \left[\sigma^+_{m_I}(\nu) - \sigma^-_{m_I}(\nu)\right] \frac{\mathrm{d}\nu}{\nu} = \pi m_I \left(\frac{\mu}{I} - \frac{Ze}{M}\right)^2 \tag{114}$$

The variables are defined as follows:

1. I_{GDH}: GDH integral with the integrand evaluated at $Q^2 = 0$
2. I: spin of the target
3. m_I: expectation value of the target spin $\vec{I}$ projected along the photon propagation axis $\hat{q} = \hat{z}$.
4. $\sigma^+_{m_I}$: total cross section for a photon with positive helicity scattering from a target in the m_I state
5. $\sigma^-_{m_I}$: total cross section for a photon with negative helicity scattering from a target in the m_I state
6. ν: lab energy of the incident (and forward scattered) *real* photon
7. μ: magnetic moment of the target
8. e: elementary unit of charge
9. Z: charge of the target in units of e
10. M: mass of the target

Note that total cross section is larger when the photon helicity and the z projection of the target spin have the same sign. This corresponds to the case when the photon and target spins are parallel. When the spins are antiparallel, signs are different. We can therefore rewrite the integrand in terms of $\sigma^{P(A)}$, the total cross section for scattering a photon scattering from target with spins parallel (antiparallel):

$$I_{\mathrm{GDH}} = \int_0^{\infty} \left[\sigma^P_{m_I}(\nu) - \sigma^A_{m_I}(\nu)\right] \frac{\mathrm{d}\nu}{\nu} = \pi \left|m_I\right| \left(\frac{\mu}{I} - \frac{Ze}{M}\right)^2 \tag{115}$$

where now we take the *absolute value* of m_I, the z projection of the target spin. This can be understood be considering table (1):

$$\sigma^P_{|m_I|} = \sigma^+_{+|m_I|} = \sigma^-_{-|m_I|} \tag{116}$$

$$\sigma^A_{|m_I|} = \sigma^-_{+|m_I|} = \sigma^+_{-|m_I|} \tag{117}$$

Using the helicity to the label the cross sections, the r.h.s of the sum rule can be expressed in terms of the anomalous magnetic moment defined in one of following three ways (in D.T.U.):

$$\mu = g\mu_N I = 2\left(Z\frac{M_{\mathrm{P}}}{M} + \kappa\right)\mu_{\mathrm{N}} I = (Z + \kappa^\star)\,\frac{e}{M} I = (2ZI + \kappa_{\mathrm{Ji}})\,\frac{e}{2M} \tag{118}$$

$$\left(\frac{\mu}{I} - \frac{Ze}{M}\right) = e\frac{\kappa}{M_P} = e\frac{\kappa^\star}{M} = e\frac{\kappa_{\mathrm{Ji}}}{2IM} \tag{119}$$

m_I value	$\|m_I\|$	$-\|m_I\|$
target spin z projection	$\Uparrow$	$\Downarrow$
positive (+) photon helicity	$\uparrow$	$\uparrow$
negative (−) photon helicity	$\downarrow$	$\downarrow$
parallel cross section, σ^P	σ^+	σ^-
antiparallel cross section, σ_A	σ^-	σ^+
+ helicity cross section, σ^+	σ^P	σ^A
− helicity cross section, σ^-	σ^A	σ^P

Table 1: For $m_I = |m_I|$, the positive (negative) helicity cross section is the parallel (antiparallel) cross section, whereas for $m_I = -|m_I|$, the negative (positive) helicity cross section is the parallel (antiparallel) cross section.

Identifying $\alpha = \frac{e^2}{4\pi}$ in (D.T.U.), noting that $g \equiv \frac{\mu}{\mu_{\rm N} I}$ and labeling the integal by m_I, this gives three equivalent forms for the sum rule:

$$I^{m_I}_{\rm GDH} = \frac{4\pi^2\alpha m_{\rm I}}{M_P^2}\kappa^2 \qquad \kappa \equiv \frac{g}{2} - Z\frac{M_P}{M} \tag{120}$$

$$I^{m_I}_{\rm GDH} = \frac{4\pi^2\alpha m_{\rm I}}{M^2}\kappa^{\star 2} \qquad \kappa^\star \equiv \frac{g}{2}\left(\frac{M}{M_P}\right) - Z \tag{121}$$

$$I^{m_I}_{\rm GDH} = \frac{\pi^2\alpha m_{\rm I}}{M^2}\left(\frac{\kappa_{\rm Ji}}{I}\right)^2 \qquad \kappa_{\rm Ji} \equiv gI\frac{M}{M_P} - 2ZI \tag{122}$$

6.7 Summing over Target Spin States

In the previous section, the GDH sum rule was written for the target in the m_I spin state. X.Ji and Y.Li [Phys. Lett. B591, 76 (2004)] performed a multipole expansion of the Compton scattering amplitude similar to the earlier work A. Pais and S.Saito. Consequently Ji & Li considered the following integral sum for a circularly polarized photon beam with positive helicity:

$$I_{\rm JL} \equiv \sum_{m_I=-I}^{+I}\int_0^\infty m_I\sigma^+_{m_I}\frac{{\rm d}\nu}{\nu} = -\left(\sum_{m_I=-I}^{+I}\int_0^\infty m_I\sigma^-_{m_I}\frac{{\rm d}\nu}{\nu}\right) \tag{123}$$

where the second sum follows from equations (116) & (117). Half the difference gives:

$$\begin{aligned} I_{\rm JL} &= \frac{1}{2}\left(\sum_{m_I=-I}^{+I}\int_0^\infty m_I\sigma^+_{m_I}\frac{{\rm d}\nu}{\nu}\right) - \frac{1}{2}\left(\sum_{m_I=-I}^{+I}\int_0^\infty m_I\sigma^-_{m_I}\frac{{\rm d}\nu}{\nu}\right) && (124)\\ &= \sum_{m_I=-I}^{+I}\left[\frac{m_I}{2}\left(\int_0^\infty\left[\sigma^+_{m_I} - \sigma^-_{m_I}\right]\frac{{\rm d}\nu}{\nu}\right)\right] && (125)\\ &= \sum_{m_I=-I}^{+I}\left[\frac{m_I}{2}I^{m_I}_{\rm GDH}\right] && (126)\\ &= \sum_{m_I=-I}^{+I}\left[\frac{m_I}{2}\frac{\pi^2\alpha m_{\rm I}}{M^2}\left(\frac{\kappa_{\rm Ji}}{I}\right)^2\right] && (127)\\ &= \frac{\pi^2\alpha}{2M^2}\left(\frac{\kappa_{\rm Ji}}{I}\right)^2\left[\sum_{m_I=-I}^{+I} m_I^2\right] && (128) \end{aligned}$$

The bracketed sum can be calculated:

$$\sum_{m_I=-I}^{+I} m_I^2 = \sum_{n=0}^{2I}(n-I)^2 = \left[\sum_{n=0}^{2I}n^2\right] - 2I\left[\sum_{n=0}^{2I}n\right] + I^2\left[\sum_{n=0}^{2I}1\right] \tag{129}$$

Symbol	^{1}H	^{1}n	^{2}H	^{3}He	^{3}H
Z	$+1$	0	$+1$	$+2$	$+1$
I	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
m	$\pm\frac{1}{2}$	$\pm\frac{1}{2}$	$0, \pm 1$	$\pm\frac{1}{2}$	$\pm\frac{1}{2}$
$M(M_P)$	1.0	1.00137841870	1.99900750082	2.9931526671	2.993717
$\mu(\mu_N)$	$+2.792847351$	-1.91304273	$+0.8574382329$	-2.127497723	$+2.9789623$
g	$+5.585694701$	-3.82608546	$+0.8574382329$	-4.254995446	$+5.9579246$
κ	$+1.792847$	-1.913043	-0.0715291	-2.795690	$+2.6449$
$\kappa^\star$	$+1.792847$	-1.915680	-0.142987	-8.367925	$+7.9182$
$\kappa_{\rm Ji}$	$+1.792847$	-1.915680	-0.285975	-8.367925	$+7.9182$
$I^m_{\rm GDH}(\mu$b$)$	± 204.78	± 233.16	$0, \pm 0.65194$	± 497.95	± 445.70

Table 2: GDH Sum Rule evaluated for the proton, neutron, deuteron, helion, and triton. All Z & I data from *CRC Handbook of Chemistry and Physics, 75th Ed.* edited by D.R. Lide (1995). Except for triton, all M, μ, & g data from CODATA 2002 [P.J. Mohr & B.N. Taylor, *RMP* **77**, 1 (2005)] from NIST website for Fundamental Physical Constants [http://physics.nist.gov/cuu/Constants/index.html]. For triton data reference information, see appendix A.

QUANTITY	VALUE	UNITS	DEFINITION
α	$(137.03599911)^{-1}$	unitless	fine structure constant
M_p	938.272029	MeV/c^2	mass of proton
M_n	939.565360	MeV/c^2	mass of neutron
M_d	1875.61282	MeV/c^2	mass of deuteron
M_3	2808.39142	MeV/c^2	mass of helion
c	299792458	m/s	speed of light in vaccum
ϵ_0	$10^7/\left(4\pi c^2\right)$	F/m	electric constant
$\hbar$	$1.05457168 \times 10^{-34}$	J $\cdot$ s	Planck's constant over 2π
$\hbar c$	197.326968	MeV $\cdot$ fm	conversion constant
1 b	100	fm^2	barn, unit of cross section
1 eV	$1.60217653 \times 10^{-19}$	J	electron-volt, unit of energy

Table 3: A bunch of relevant constants and stuff are listed above. All numbers are CODATA 2002 via NIST website. [see caption for table (2) for full citation]

$$= \frac{2I(2I+1)(4I+1)}{6} - 2I\frac{2I(2I+1)}{2} + I^2\,(2I+1) \tag{130}$$

$$= \frac{I}{3}\left[8I^2 + 6I + 1 - 12I^2 - 6I + 6I^2 + 3I\right] \tag{131}$$

$$= \frac{I}{3}\left[2I^2 + 3I + 1\right] \tag{132}$$

$$= \frac{I}{3}(I+1)(2I+1) \tag{133}$$

This gives a modified version of the GDH sum rule which is equivalent to equations (95) & (96):

$$I_{\rm JL} = \sum_{m_I=-I}^{+I} \int_0^\infty m_I \sigma^+_{m_I} \frac{{\rm d}\nu}{\nu} = \frac{\pi^2\alpha}{6M^2}\left[\frac{(I+1)(2I+1)}{I}\right]\kappa^2_{\rm Ji} \tag{134}$$

7 Evaluating the GDH Sum Rule

To actually calculate the sum rule, one has to convert the sum into measured units, which amounts to putting the factors of $\hbar$ and c back into the equation. The l.h.s. of the sum rule evaluates to a quantity that has the units of a cross section or distance squared. The only quantity on the r.h.s. of the sum rule that is not dimensionless is the mass squared. Therefore some combination of $\hbar$, c, and mass squared needs to result in distance squared:

$$\frac{[\hbar]^2}{[M]^2[c]^2} = \frac{[\hbar]^2 c^2}{[Mc^2]^2} = \frac{\mathrm{J}^2\mathrm{s}^2\frac{\mathrm{m}^2}{\mathrm{s}^2}}{\mathrm{J}^2} = \mathrm{m}^2 \tag{135}$$

Putting these in these scale factors:

$$I_{\mathrm{GDH}}^{m_I} = \frac{4\pi^2\alpha m_{\mathrm{I}}\hbar^2}{M_P^2c^2}\kappa^2 \qquad \kappa \equiv \frac{g}{2} - Z\frac{M_P}{M} \tag{136}$$

$$I_{\mathrm{GDH}}^{m_I} = \frac{4\pi^2\alpha m_{\mathrm{I}}\hbar^2}{M^2c^2}\kappa^{\star 2} \qquad \kappa^\star \equiv \frac{g}{2}\left(\frac{M}{M_P}\right) - Z \tag{137}$$

$$I_{\mathrm{GDH}}^{m_I} = \frac{\pi^2\alpha m_{\mathrm{I}}\hbar^2}{M^2c^2}\left(\frac{\kappa_{\mathrm{Ji}}}{I}\right)^2 \qquad \kappa_{\mathrm{Ji}} \equiv gI\frac{M}{M_P} - 2ZI \tag{138}$$

It is usefull to express the sum rule in terms of the Gerasimov cross section defined by:

$$\sigma_G \equiv \frac{2\pi^2\alpha\hbar^2}{M_P^2c^2} = 63.7104\ \mu\mathrm{b} \tag{139}$$

This gives the following relations:

$$I_{\mathrm{GDH}}^{m_I} = 2\sigma_G m_{\mathrm{I}}\kappa^2 \qquad \kappa \equiv \frac{g}{2} - Z\frac{M_P}{M} \tag{140}$$

$$I_{\mathrm{GDH}}^{m_I} = \frac{2\sigma_G m_{\mathrm{I}}}{\left(\frac{M}{M_P}\right)^2}\kappa^{\star 2} \qquad \kappa^\star \equiv \frac{g}{2}\left(\frac{M}{M_P}\right) - Z \tag{141}$$

$$I_{\mathrm{GDH}}^{m_I} = \frac{2\sigma_G m_{\mathrm{I}}}{\left(\frac{M}{M_P}\right)^2}\left(\frac{\kappa_{\mathrm{Ji}}}{2I}\right)^2 \qquad \kappa_{\mathrm{Ji}} \equiv gI\frac{M}{M_P} - 2ZI \tag{142}$$

The GDH sum rule for real circularly polarized photons incident on a spin I target in SI units:

$$I^m_{\rm GDH} = \int_0^\infty \left[\sigma^+_m(\nu) - \sigma^-_m(\nu)\right] \frac{\mathrm{d}\nu}{\nu} = \pi^2\alpha m \left(\frac{\mu - \mu_{\rm point}}{M_P \mu_N I}\right)^2 \tag{143}$$

$$= 2\sigma_G m \kappa^2 = \frac{2\sigma_G m}{M^2}\kappa^{\star 2} = \frac{\sigma_G m}{2M^2 I^2}\kappa^2_{\rm Ji} \tag{144}$$

where the following quantities are calculated by:

$$\mu = g\mu_N I \tag{145}$$

$$\mu_{\rm point} \equiv \frac{2Z}{M}\mu_{\rm N} I \tag{146}$$

$$\sigma_G \equiv \frac{2\pi^2\alpha\hbar^2}{M_P^2 c^2} = 63.7104\ \mu\text{b} = 0.1636204\ \text{GeV}^{-2} = 0.00637104\ \text{fm}^{-2} \tag{147}$$

$$\kappa \equiv \frac{g}{2} - \frac{Z}{M} \tag{148}$$

$$\kappa^\star \equiv \frac{g}{2}M - Z \tag{149}$$

$$\kappa_{\rm Ji} \equiv gIM - 2ZI \tag{150}$$

and all of the quantities are defined below:

Z**:** charge of the target in units of e

M**:** mass of the target in units of the proton mass M_P

M_P**:** proton mass in SI units

I**:** spin of the target

μ**:** magnetic moment of the target in SI units

$\mu_{\rm point}$**:** magnetic moment of the target *if* it were an ideal point-like object in SI units

μ_N**:** nuclear magnetons in SI units

g**:** Landé g-factor

σ_G**:** Gerasimov cross section, defined by equation (147)

m**:** projection of the target spin $\vec{I}$ onto the photon propagation axis

$\kappa, \kappa^\star, \kappa_{\rm Ji}$**:** anomalous (part of the the) magnetic moment of the target defined in various ways

ν**:** lab energy of the incident (and scattered) photon

σ^+_m**:** total cross section for scattering positive helicity photons

σ^-_m**:** total cross section for scattering negative helicity photons

μ_t/μ_p	uncertainty	Authors	Citation
1.06666	0.00010	H.L. Anderson A. Novick	Phys. Rev. 71, 372 (1947)
1.067 1.066636	0.001 0.00001	F. Bloch A.C. Graves M. Packard R.W. Spence	Phys. Rev. 71, 373 (1947) Phys. Rev. 71, 551 (1947)
1.06663986	0.00000011	W. Duffy, Jr.	Phys. Rev. 115, 1012 (1959)

Table 4: List of various measurements of the relative triton magnetic moment.

8 Extended GDH Integrals

I_A, I_B, I_C, and all that photon flux factor (Hand, Gilman, etc.) stuff

9 Extended GDH Sum Rules ($Q^2 > 0$)

That Ji & Osbourne Paper

10 Chiral Slopes at $Q^2 = 0$

11 The MAID Model

12 Review of Experimental Efforts

12.1 real photon GDH - Mainz

12.2 High Q^2 GDH - Hermes

12.3 JLab Mid Q^2 Neutron

E94010

12.4 JLab Low Q^2 Neutron

E97110, Vince and me

12.5 JLab Mid Q^2 Proton/Deuteron

Renee Fatemi and Yelena Prok

12.6 JLab Low Q^2 Proton/Deuteron

Josh Pierce and John Mellor

12.7 HIGS at TUNL

A Triton Data

A.1 Spin & Magnetic Moment

A casual search through PROLA yields four papers describing measurements of the magnetic moment of the triton (tritium nucleus). The results are summarized in table (4). All of the measurements compared the resonance frequencies of the triton to the proton. These measurements occured while the triton and proton were in atoms, meaning the nuclear charge was shielded. CODATA 2002 shows that a shielded proton has a magnetic moment that is smaller by about 26 ppm. The shielding should be similar for both the triton and proton, therefore this effect should cancel to first order. A wieghted fit to the data in the table multiplied by the proton magnetic moment gives the triton magnetic moment in nuclear magnetons:

$$\frac{\mu_t}{\mu_N} = 2.9789623 \tag{151}$$

In addition, the Los Alamos group argued that the spin of the triton has to be $\frac{1}{2}$ on the basis of the ratio of the size of the two signals. F. Bloch had earlier derived in his famous "Nuclear Induction" paper [Phys. Rev. 70, 460 (1946)] an equation estimating the size of signal induced by nuclear spins in thermal equilibrium [equation (29)]:

$$V = \pm NAn\frac{j(j+1)}{3kT}h^2\gamma\omega^2\frac{\cos\omega t}{1+\delta^2}^{\frac{1}{2}} \tag{152}$$

where N is the number of turns in the pick up coil, A is the area of the pick up coils, n is the density of the nuclear spins, j is the spin, γ is the gyromagnetic ratio, ω is the frequency of the rf field, and δ is the detuning from resonance. In both of the Bloch,*et al* measurements the ratio of the size of the signals was approximately the ratio of the densities of tritium and hydrogen which indicated that the nuclear spins were the same. E.B. Nelson & J.E. Nafe [Phys. Rev. 75, 1194 (1948)] verified that the triton spin was $\frac{1}{2}$ by observing the hyperfine structure of tritium using the atomic beam resonance methods popular during that time at Columbia University.

A.2 Mass ratio with proton

The mass of the triton (just the nucleus) itself is difficult to find. The tritium mass is easily found but includes the effect of a bound electron. Given known masses for the deuteron, helion, alpha particle (^4He nucleus), deuterium, helium-3, and helium-4 (see table (5)) an estimate for the triton mass can be found. It is assumed that an atomic mass can be written as a sum of its constituent masses (m_p, m_n, m_e), binding energies (b_n, b_e), and repulsion energies (r_n, r_e):

$$A = \underbrace{pm_p + nm_n + \overbrace{f(n,p)b_n + g(n,p)r_n}^{\delta N}}_{\text{nuclear mass}} + \underbrace{pm_e + \overbrace{j(n,p)b_e + k(n,p)r_e}^{\delta A}}_{\text{electrical mass}} \tag{153}$$

where f, g, j, k are some unknown functions of proton and neutron numbers p, n.

Obviously, the nuclear binding effects have a much greater effect on the atomic mass than the electrical mass. Because of isospin symmetry, one would expect that the binding strong force between nucleons is the same. Therefore the total effect of the strong binding force should scale with the distinct number of possible nucleon pairs. This gives the following guess for $f(n,p) = {}_{n+p}C_2 = (n+p)(n+p-1)/2$. The electrostatic repulsion of the protons should scale with the number of distinct number of proton pairs. We'll assume that the neutrons contribute to the electrostatic interaction within the nucleus by some shielding to the proton repulsion. This gives the following guess for $g(n,p) = (1+sn)\ {}_pC_2 = (1+sn)p(p-1)/2$. Putting all this together, our guess for the functional form of δN is:

$$\delta N = f(n,p)b_n + g(n,p)r_p \approx (n+p)(n+p-1)\frac{b_n}{2} + (1-s_0 n)(p-1)p\frac{r_p}{2} \tag{154}$$

electron	$5.4857990945 \times 10^{-4}$	neutron	1.00866491560	δA(ppm)	δN(ppm)
hydrogen	1.0078250321	proton	1.00727646688	-0.0146895	0.0
deuterium	2.0141017780	deuteron	2.01355321270	-0.0146095	-2388.16978
tritium	3.0160492675	triton	3.01550070212	-0.0145295*	-9105.59596*
helium-3	3.0160293097	helion	3.0149322434	-0.0935189	-8285.60596
helium-4	4.0026032497	α-particle	4.001506179149	-0.0892589	-30376.58582

Table 5: List of relevant masses in relative atomic mass units. All values from NIST website, except *-values which are calculated in the text.

where b_n is the average binding between two nucleons, r_p is the average electrostatic repulsion between the protons, and s_0 is the sheilding of the proton charge by the neutron. One good sign is that this formula predicts $\delta N = 0$ for the nucleus of the hydrogen atom. We have 3 free parameters and 4 atoms of known values. However ^4He has a "magic number" of nucleons and therefore is very stable and almost certainly does not follow from the previous arguments. If we assume that the triton mass is equal to the helion mass plus the neutron-proton mass difference, then the value for δA_t is wildly inconsistent with known values for other atoms:

$$\delta N_t \approx \delta N_3 \quad (155)$$
$$m_t \approx m_3 - m_p + m_n = 3.01632069212 \quad (156)$$
$$\delta A_t \approx A_t - m_t - m_e = -820.00453 \text{ (ppm)} \quad (157)$$

On the other hand, δA is quite small and fairly constant for hydrogen and deuterium. Therefore an estimate for the triton mass can be gotten by the following:

$$\delta A_t \approx 2\delta A_d - \delta A_1 = -0.0145295 \text{ (ppm)} \quad (158)$$
$$m_t \approx A_t - m_e - \delta A_t = 3.01550070212 \quad (159)$$
$$\delta N_t \approx m_t - m_p - 2m_n = -9105.59596 \text{ (ppm)} \quad (160)$$

which gives the triton to proton mass ratio:

$$\frac{m_t}{m_p} = M_t = 2.993717 \quad (161)$$