Magnet Box Coils Current Model

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Abstract

Presented here is a simple minded model to explain the apparently non-linear relationship between the control voltage input to the coils power supply and the readback field in the box.

Contents

1 Introduction

A field is produced inside a "magnet box" made of iron by a set of coils. These coils are powered by a power supply. The output of the power supply is controlled by an external control signal. An ammeter is placed in series between the coils and the power supply which reads back the current. A gaussmeter is placed inside the box near the center to measure the magnetic field.

It has been observed that the measured field during a linear triangular voltage ramp of the power supply does not result in a linear triangular field. There appears to be a lag between the control voltage and the measured field. The size of the lag and the amount of deviation from a linear triangular field ramp both seem to depend on the ramp rate of the control voltage. If the control voltage is incrementally increased in discrete steps and if at each step there is a relatively long dwell time used to allow for the field measurement to settle, then the amount of deviation from a linear triangular field ramp appears to be minimized as well. Regardless of the type of control voltage ramping, the field eventually returns to the same value it had just before the start of the field ramp.

Based on these observations, we hypothesize that the slightly (but sufficiently) non-linear behaviour of the field is due to a relatively long reponse time of the coils due to inductive effects. Qualitatively, this may be reasonable because the presence of iron inside the coil volume increases the inductance of the coils. Iron has a DC magnetic permeability that is on order of 100 to 1000 times greater than the vacuum magnetic permeability.

Therefore, in this model, we assume:

1. There are no hysteresis effects due to the iron in the box.

2. The field in the box is strictly linearly proportional to the current in the coils:

$$\vec{B} = I\vec{f}(\vec{r}) + \vec{B}_{\mathrm{E}} + \vec{B}_{\mathrm{A}} \tag{1}$$

where $\vec{B}_{\mathrm{E}}$ is the Earth's field, $\vec{B}_{\mathrm{A}}$ is the field due to other sources in Hall A (such as the HRS or Bigbite), I is the current in the coils, and $\vec{f}(\vec{r})$ is a geometric factor.

3. The conversion coefficient between the control voltage and the resulting current in the coils is described by an "effective resistance," R.

4. The response time constant is parameterized by τ.

5. The field and current measurement time scales are much faster than both the field sweep and the field response time.

2 Effect of Inductance

We'll assume that we have just a series LR circuit with a voltage source. This gives:

$$V(t) = I(t)R + L\frac{dI}{dt} \tag{2}$$

We'll define the time constant as:

$$\tau = \frac{L}{R} \tag{3}$$

We'll use the ansatz that the current can be written as a product of two functions:

$$\begin{aligned} I(t) &= f(t)g(t) \qquad (4) \\ I' &= fg' + f'g \qquad (5) \end{aligned}$$

Rewriting our original differential equation and dividing by $I(t)$:

$$\begin{aligned} \frac{V}{R} &= fg + \tau\left(f'g + fg'\right) \qquad (6) \\ \frac{V}{fgR} &= 1 + \tau\frac{f'}{f} + \tau\frac{g'}{g} \qquad (7) \end{aligned}$$

Now we make the choice that:

$$\tau\frac{g'}{g} + 1 = 0 \tag{8}$$

Which gives

$$\begin{aligned} g' &= -\frac{g}{\tau} \qquad (9) \\ g(t) &= g(0)\exp\left(-\frac{t}{\tau}\right) \qquad (10) \end{aligned}$$

and therefore results in:

$$\begin{aligned} \frac{V}{fgR} &= \tau\frac{f'}{f} \qquad (11) \\ f' &= \frac{V}{gR\tau} \qquad (12) \end{aligned}$$

which can be solved by direct intergation using the dummy variable u:

$$\frac{df}{du} = \frac{V(u)}{g(u)R\tau} \tag{13}$$

$$f(u) = \frac{1}{R\tau}\int \frac{V(u)}{g(u)}du \tag{14}$$

$$= \frac{1}{R\tau g(0)}\int V(u)\exp\left(\frac{u}{\tau}\right)du \tag{15}$$

Therefore, given a time dependant supply voltage of $V(t)$ yields a current:

$$I(t) = g(t)f(t) \tag{16}$$

$$= g(0)\exp\left(-\frac{t}{\tau}\right)\frac{1}{R\tau g(0)}\int V(u)\exp\left(\frac{u}{\tau}\right)du \tag{17}$$

$$= \int_{-\infty}^{t}\frac{V(u)}{R}\exp\left(\frac{u-t}{\tau}\right)\frac{du}{\tau} \tag{18}$$

Note that since $-\infty < u \le t$, the quantity $u - t$ will always be less than or equal to zero. We'll quickly verify that $I(t)$ really does satisfy the differential equation using Leibnitz's Rule:

$$\frac{dI}{dt} = \left(\frac{-1}{\tau}\right)\int_{-\infty}^{t}\frac{V(u)}{R}\exp\left(\frac{u-t}{\tau}\right)\frac{du}{\tau} + \frac{V(t)}{R}\frac{1}{\tau} \tag{19}$$

$$= -\frac{I}{\tau} + \frac{V}{\tau R} \tag{20}$$

$$\frac{V}{R} = I + \tau I' \tag{21}$$

3 Helium-3 Field Sweep - Triangluar Voltage Ramp

A field sweep is typically generated by a voltage ramp:

$$V(t) = \left\{ \begin{array}{cc} V_1 = V_0 & t \le -T \\ V_2 = V_0 + V_A\left(1 + \frac{t}{T}\right) & -T \le t \le 0 \\ V_3 = V_0 + V_A\left(1 - \frac{t}{T}\right) & 0 \le t \le +T \\ V_4 = V_0 & t \ge +T \end{array} \right\} \tag{22}$$

The current is therefore also a continuous piecewise function:

$$I_1 = I(t \le -T) = \int_{-\infty}^{t}\frac{V_1(u)}{R}\exp\left(\frac{u-t}{\tau}\right)\frac{du}{\tau} \tag{23}$$

$$I_2 = I(-T \le t \le 0) = \int_{-\infty}^{-T}\frac{V_1(u)}{R}\exp\left(\frac{u-t}{\tau}\right)\frac{du}{\tau} \tag{24}$$

$$+ \int_{-T}^{t}\frac{V_2(u)}{R}\exp\left(\frac{u-t}{\tau}\right)\frac{du}{\tau} \tag{25}$$

$$I_3 = I(0 \le t \le +T) = \int_{-\infty}^{-T}\frac{V_1(u)}{R}\exp\left(\frac{u-t}{\tau}\right)\frac{du}{\tau} \tag{26}$$

$$+ \int_{-T}^{0}\frac{V_2(u)}{R}\exp\left(\frac{u-t}{\tau}\right)\frac{du}{\tau} \tag{27}$$

$$+ \int_{0}^{t}\frac{V_3(u)}{R}\exp\left(\frac{u-t}{\tau}\right)\frac{du}{\tau} \tag{28}$$

$$I_4 = I(t \ge +T) = \int_{-\infty}^{-T}\frac{V_1(u)}{R}\exp\left(\frac{u-t}{\tau}\right)\frac{du}{\tau} \tag{29}$$

$$+\int_{-T}^{0} \frac{V_2(u)}{R} \exp\left(\frac{u-t}{\tau}\right) \frac{du}{\tau} \tag{30}$$

$$+\int_{0}^{+T} \frac{V_3(u)}{R} \exp\left(\frac{u-t}{\tau}\right) \frac{du}{\tau} \tag{31}$$

$$+\int_{+T}^{t} \frac{V_4(u)}{R} \exp\left(\frac{u-t}{\tau}\right) \frac{du}{\tau} \tag{32}$$

To do these integrals, first let's consider the following two integrals:

$$I_a(u) = \int \frac{V_0}{R} \exp\left(\frac{u-t}{\tau}\right) \frac{du}{\tau} \tag{33}$$

$$= \frac{V_0}{R} \exp\left(\frac{u-t}{\tau}\right) \tag{34}$$

$$I_b(u) = \int \frac{V_A u}{RT} \exp\left(\frac{u-t}{\tau}\right) \frac{du}{\tau} \tag{35}$$

$$= \frac{V_A}{RT\tau} \exp\left(\frac{-t}{\tau}\right) \int u \exp\left(\frac{u}{\tau}\right) du \tag{36}$$

$$= \frac{V_A}{RT}(u-\tau) \exp\left(\frac{u-t}{\tau}\right) \tag{37}$$

For the first time section:

$$I_1(t) = \int_{-\infty}^{t} \frac{V_0}{R} \exp\left(\frac{u-t}{\tau}\right) \frac{du}{\tau} \tag{38}$$

$$= \frac{V_0}{R} \exp\left(\frac{t-t}{\tau}\right) - \frac{V_0}{R} \exp\left(\frac{-\infty-t}{\tau}\right) \tag{39}$$

$$= \frac{V_0}{R} \tag{40}$$

For the second time section:

$$I_2(t) = \int_{-\infty}^{-T} \frac{V_0}{R} \exp\left(\frac{u-t}{\tau}\right) \frac{du}{\tau} + \int_{-T}^{t} \frac{V_0 + V_A\left(1+\frac{u}{T}\right)}{R} \exp\left(\frac{u-t}{\tau}\right) \frac{du}{\tau} \tag{41}$$

$$= \int_{-\infty}^{t} \frac{V_0}{R} \exp\left(\frac{u-t}{\tau}\right) \frac{du}{\tau} + \int_{-T}^{t} \frac{V_A}{R} \exp\left(\frac{u-t}{\tau}\right) \frac{du}{\tau} + \int_{-T}^{t} \frac{\frac{V_A}{T}u}{R} \exp\left(\frac{u-t}{\tau}\right) \frac{du}{\tau} \tag{42}$$

$$= \frac{1}{R}\left[V_0 + V_A\left(1+\frac{t}{T}\right)\right] - \frac{V_A\tau}{RT}\left[1-\exp\left(\frac{-T-t}{\tau}\right)\right] \tag{43}$$

For the third time section:

$$I_3(t) = \int_{-\infty}^{-T} \frac{V_0}{R} \exp\left(\frac{u-t}{\tau}\right) \frac{du}{\tau} + \int_{-T}^{0} \frac{V_0 + V_A\left(1+\frac{u}{T}\right)}{R} \exp\left(\frac{u-t}{\tau}\right) \frac{du}{\tau} \tag{44}$$

$$+\int_{0}^{t} \frac{V_0 + V_A\left(1-\frac{u}{T}\right)}{R} \exp\left(\frac{u-t}{\tau}\right) \frac{du}{\tau} \tag{45}$$

$$= \int_{-\infty}^{t} \frac{V_0}{R} \exp\left(\frac{u-t}{\tau}\right) \frac{du}{\tau} + \int_{-T}^{t} \frac{V_A}{R} \exp\left(\frac{u-t}{\tau}\right) \frac{du}{\tau} + \int_{-T}^{0} \frac{V_A\frac{u}{T}}{R} \exp\left(\frac{u-t}{\tau}\right) \frac{du}{\tau} \tag{46}$$

$$-\int_{0}^{t} \frac{V_A\frac{u}{T}}{R} \exp\left(\frac{u-t}{\tau}\right) \frac{du}{\tau} \tag{47}$$

$$= \frac{V_0}{R} + \frac{V_A}{R} - \frac{V_A}{R} \exp\left(\frac{-T-t}{\tau}\right) - \frac{V_A\tau}{RT} \exp\left(-\frac{t}{\tau}\right) + \frac{V_A}{RT}(T+\tau) \exp\left(\frac{-T-t}{\tau}\right) \tag{48}$$

$$-\frac{V_A}{RT}(t-\tau) - \frac{V_A\tau}{RT} \exp\left(-\frac{t}{\tau}\right) \tag{49}$$

$$= \frac{1}{R}\left[V_0 + V_A\left(1-\frac{t}{T}\right)\right] + \frac{V_A\tau}{RT}\left[1 - 2\exp\left(-\frac{t}{\tau}\right) + \exp\left(\frac{-T-t}{\tau}\right)\right] \tag{50}$$

For the last time section:

$$\begin{aligned}
I_4(t) &= \int_{-\infty}^{-T} \frac{V_0}{R}\exp\left(\frac{u-t}{\tau}\right)\frac{du}{\tau} + \int_{-T}^{0} \frac{V_0 + V_A\left(1+\frac{u}{T}\right)}{R}\exp\left(\frac{u-t}{\tau}\right)\frac{du}{\tau} && (51)\\
&\quad + \int_{0}^{+T} \frac{V_0 + V_A\left(1-\frac{u}{T}\right)}{R}\exp\left(\frac{u-t}{\tau}\right)\frac{du}{\tau} + \int_{+T}^{t} \frac{V_0}{R}\exp\left(\frac{u-t}{\tau}\right)\frac{du}{\tau} && (52)\\
&= \int_{-\infty}^{t} \frac{V_0}{R}\exp\left(\frac{u-t}{\tau}\right)\frac{du}{\tau} + \int_{-T}^{+T} \frac{V_A}{R}\exp\left(\frac{u-t}{\tau}\right)\frac{du}{\tau} && (53)\\
&\quad + \int_{-T}^{0} \frac{V_A u}{RT}\exp\left(\frac{u-t}{\tau}\right)\frac{du}{\tau} - \int_{0}^{+T} \frac{V_A u}{RT}\exp\left(\frac{u-t}{\tau}\right)\frac{du}{\tau} && (54)\\
&= \frac{V_0}{R} + \frac{V_A\tau}{RT}\left[\exp\left(\frac{T-t}{\tau}\right) - 2\exp\left(-\frac{t}{\tau}\right) + \exp\left(\frac{-T-t}{\tau}\right)\right] && (55)
\end{aligned}$$

Therefore we can rewrite the current as the sum of the instantaneous reponse and a term proportional to the "lag time" function $l(t)$:

$$I(t) = \frac{V(t)}{R} + \frac{V_A\tau}{RT}l(t) \tag{56}$$

where the deviations from an ideal triangular ramp are parameterized by the "lag time" function:

$$l(t) = \left\{ \begin{array}{cc} 0 & t \leq -T \\ -1 + \exp\left(\frac{-T-t}{\tau}\right) & -T \leq t \leq 0 \\ 1 - 2\exp\left(-\frac{t}{\tau}\right) + \exp\left(\frac{-T-t}{\tau}\right) & 0 \leq t \leq +T \\ \exp\left(\frac{T-t}{\tau}\right) - 2\exp\left(-\frac{t}{\tau}\right) + \exp\left(\frac{-T-t}{\tau}\right) & t \geq +T \end{array} \right\} \tag{57}$$

See figure (1) for a graphical depiction.

4 Field Change - Unit Step Function

We'll assume that a change of voltage is instantaneous relative to the change in the current. Therefore we can write the voltage as:

$$V(t) = \left\{ \begin{array}{cc} V_0 & t \leq 0 \\ V_0 + V_A & t \geq 0 \end{array} \right\} \tag{58}$$

The current is therefore:

$$\begin{aligned}
I_1 = I(t \leq 0) &= \int_{-\infty}^{0} \frac{V_0}{R}\exp\left(\frac{u-t}{\tau}\right)\frac{du}{\tau} && (59)\\
I_2 = I(t > 0) &= \int_{-\infty}^{0} \frac{V_0}{R}\exp\left(\frac{u-t}{\tau}\right)\frac{du}{\tau} + \int_{0}^{t} \frac{V_0 + V_A}{R}\exp\left(\frac{u-t}{\tau}\right)\frac{du}{\tau} && (60)
\end{aligned}$$

Using the integrals calculated before, this gives:

$$\begin{aligned}
I_1 &= \frac{V_0}{R} && (61)\\
I_2 &= \frac{V_0 + V_A}{R} - \frac{V_A}{R}\exp\left(-\frac{t}{\tau}\right) && (62)
\end{aligned}$$

Therefore we can rewrite the current as the sum of the instantaneous reponse and a term proportional to the "lag time" function $l(t)$:

$$I(t) = \frac{V(t)}{R} + \frac{V_A}{R}l(t) \tag{63}$$

where the deviations from an ideal unit step fucntion are parameterized by the "lag time" function:

$$l(t) = \left\{ \begin{array}{cc} 0 & t \leq 0 \\ -\exp\left(-\frac{t}{\tau}\right) & t \geq 0 \end{array} \right\} \tag{64}$$

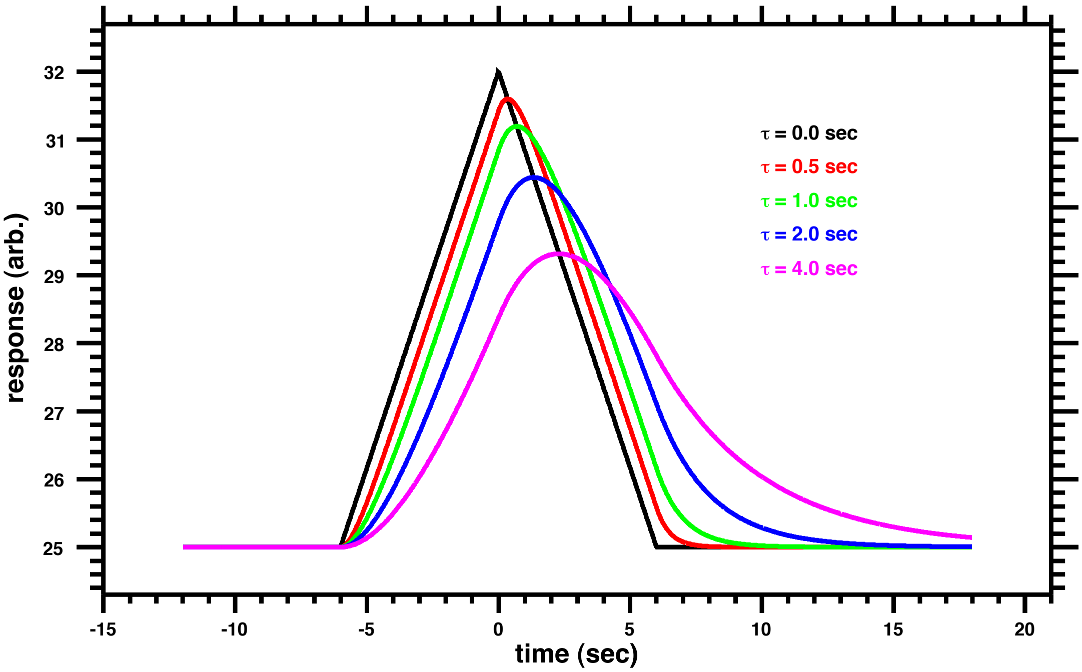

Figure 1: Triangular Sweep for $V_0/R = 25.0, V_A/R = 7.0, T = 6.0$ and $\tau = 0.5, 1.0, 2.0, 4.0$

5 Water Field Sweep - Trapezoidal Voltage Ramp

A field sweep is typically generated by a voltage ramp:

$$V(t) = \left\{ \begin{array}{cc} V_1 = V_0 & t \leq -T \\ V_2 = V_0 + V_A \left(1 + \frac{t}{T}\right) & -T \leq t \leq 0 \\ V_3 = V_0 + V_A & 0 \leq t \leq T_W \\ V_4 = V_0 + V_A \left(1 - \frac{(t-T_W)}{T}\right) & T_W \leq t \leq T + T_W \\ V_5 = V_0 & t \geq T + T_W \end{array} \right\} \tag{65}$$

The current is therefore also a continuous piecewise function:

$$I_1 = I(t \leq -T) = \int_{-\infty}^{t} \frac{V_1(u)}{R} \exp\left(\frac{u-t}{\tau}\right) \frac{du}{\tau} \tag{66}$$

$$I_2 = I(-T \leq t \leq 0) = \int_{-\infty}^{-T} \frac{V_1(u)}{R} \exp\left(\frac{u-t}{\tau}\right) \frac{du}{\tau} \tag{67}$$

$$+ \int_{-T}^{t} \frac{V_2(u)}{R} \exp\left(\frac{u-t}{\tau}\right) \frac{du}{\tau} \tag{68}$$

$$I_3 = I(0 \leq t \leq T_W) = \int_{-\infty}^{-T} \frac{V_1(u)}{R} \exp\left(\frac{u-t}{\tau}\right) \frac{du}{\tau} \tag{69}$$

$$+ \int_{-T}^{0} \frac{V_2(u)}{R} \exp\left(\frac{u-t}{\tau}\right) \frac{du}{\tau} \tag{70}$$

$$+ \int_{0}^{t} \frac{V_3(u)}{R} \exp\left(\frac{u-t}{\tau}\right) \frac{du}{\tau} \tag{71}$$

$$I_4 = I(T_W \leq t \leq T_W + T) = \int_{-\infty}^{-T} \frac{V_1(u)}{R} \exp\left(\frac{u-t}{\tau}\right) \frac{du}{\tau} \tag{72}$$

$$+\int_{-T}^{0} \frac{V_2(u)}{R} \exp\left(\frac{u-t}{\tau}\right) \frac{du}{\tau} \tag{73}$$

$$+\int_{0}^{T_W} \frac{V_3(u)}{R} \exp\left(\frac{u-t}{\tau}\right) \frac{du}{\tau} \tag{74}$$

$$+\int_{T_W}^{t} \frac{V_4(u)}{R} \exp\left(\frac{u-t}{\tau}\right) \frac{du}{\tau} \tag{75}$$

$$I_5 = I(t \geq T_W + T) \quad = \quad \int_{-\infty}^{-T} \frac{V_1(u)}{R} \exp\left(\frac{u-t}{\tau}\right) \frac{du}{\tau} \tag{76}$$

$$+\int_{-T}^{0} \frac{V_2(u)}{R} \exp\left(\frac{u-t}{\tau}\right) \frac{du}{\tau} \tag{77}$$

$$+\int_{0}^{T_W} \frac{V_3(u)}{R} \exp\left(\frac{u-t}{\tau}\right) \frac{du}{\tau} \tag{78}$$

$$+\int_{T_W}^{T_W+T} \frac{V_4(u)}{R} \exp\left(\frac{u-t}{\tau}\right) \frac{du}{\tau} \tag{79}$$

$$+\int_{T_W+T}^{t} \frac{V_5(u)}{R} \exp\left(\frac{u-t}{\tau}\right) \frac{du}{\tau} \tag{80}$$

Using the results from the previous sections, it is straightforward to find the integral of each part. For the first time section:

$$I_1 \quad = \quad \int_{-\infty}^{t} \frac{V_0}{R} \exp\left(\frac{u-t}{\tau}\right) \frac{du}{\tau} = \frac{V_0}{R} \tag{81}$$

For the second time section:

$$I_2 \quad = \quad \frac{V_0 + V_A\left(1+\frac{t}{T}\right)}{R} + \frac{V_A \tau}{RT}\left[-1 + \exp\left(\frac{-T-t}{\tau}\right)\right] \tag{82}$$

For the third time section:

$$I_3 \quad = \quad \frac{V_0 + V_A}{R} - \frac{V_A \tau}{RT} \exp\left(-\frac{t}{\tau}\right)\left[1 - \exp\left(-\frac{T}{\tau}\right)\right] \tag{83}$$

For the fourth time section:

$$I_4 \quad = \quad \frac{V_0 + V_A\left(1+\frac{T_W - t}{T}\right)}{R} + \frac{V_A \tau}{RT}\left(1 - \exp\left(-\frac{t}{\tau}\right)\left[1 - \exp\left(-\frac{T}{\tau}\right) + \exp\left(\frac{T_W}{\tau}\right)\right]\right) \tag{84}$$

For the fifth time section:

$$I_5 \quad = \quad \frac{V_0}{R} - \frac{V_A \tau}{RT} \exp\left(-\frac{t}{\tau}\right)\left[1 - \exp\left(-\frac{T}{\tau}\right) + \exp\left(\frac{T_W}{\tau}\right) - \exp\left(\frac{T_W+T}{\tau}\right)\right] \tag{85}$$

Therefore we can rewrite the current as the sum of the instantaneous reponse and a term proportional to the "lag time" function $l(t)$:

$$I(t) = \frac{V(t)}{R} + \frac{V_A}{R} l(t) \tag{86}$$

where the deviations from an ideal unit step fucntion are parameterized by the "lag time" function:

$$l(t) = \left\{ \begin{array}{cc} 0 & t \leq -T \\ -1 + \exp\left(\frac{-T-t}{\tau}\right) & -T \leq t \leq 0 \\ -\exp\left(-\frac{t}{\tau}\right)\left[1 - \exp\left(-\frac{T}{\tau}\right)\right] & 0 \leq t \leq T_W \\ 1 - \exp\left(-\frac{t}{\tau}\right)\left[1 - \exp\left(-\frac{T}{\tau}\right) + \exp\left(\frac{T_W}{\tau}\right)\right] & T_W \leq t \leq T + T_W \\ -\exp\left(-\frac{t}{\tau}\right)\left[1 - \exp\left(-\frac{T}{\tau}\right) + \exp\left(\frac{T_W}{\tau}\right) - \exp\left(\frac{T_W+T}{\tau}\right)\right] & t \geq T + T_W \end{array} \right\} \tag{87}$$

Note that if there is no "hold" period after the initial ramp ($T_W = 0$), then this results reduces to that of the Helium triangle sweep. See figure (2) for a graphical depiction.

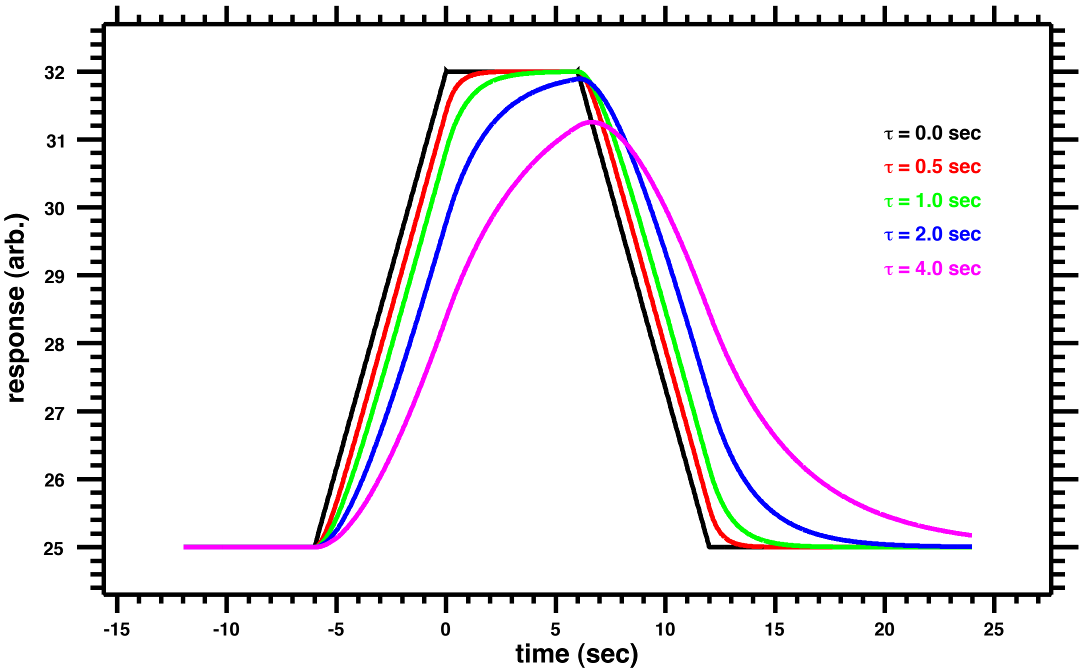

Figure 2: Trapezoidal Sweep for $V_0/R = 25.0, V_A/R = 7.0, T = 6.0, T_W = 6.0$ and $\tau = 0.5, 1.0, 2.0, 4.0$

6 Conclusions

- Although we've only shown this for a few examples, it is more generally true that the reponse of all linear piecewise excitations of a system can be written as a sum of the zero lagtime reponse and a lagtime function.

- For a triangular ramp, it is clear that the peak response is dampened and shifted in time. It is straightforward to show that the damping ratio and time offset relative to the zero lagtime system response are given by:

$$\frac{I_{\text{peak}}(\tau \neq 0) - V_0/R}{I_{\text{peak}}(\tau = 0) - V_0/R} = 1 - \frac{t_{\text{offset}}}{T} \tag{88}$$

$$\frac{t_{\text{offset}}}{T} = \frac{\tau}{T} \log\left[2 - \exp\left(-\frac{T}{\tau}\right)\right] \tag{89}$$

$$\text{for } T \gg \tau \quad \rightarrow \quad t_{\text{offset}} \approx \tau \log(2) \tag{90}$$

$$\text{for } T \ll \tau \quad \rightarrow \quad t_{\text{offset}} \approx T \tag{91}$$

See figs (3) and (4) for plots of the relative time offset as a function of the relative lagtime in the two approximation regimes.

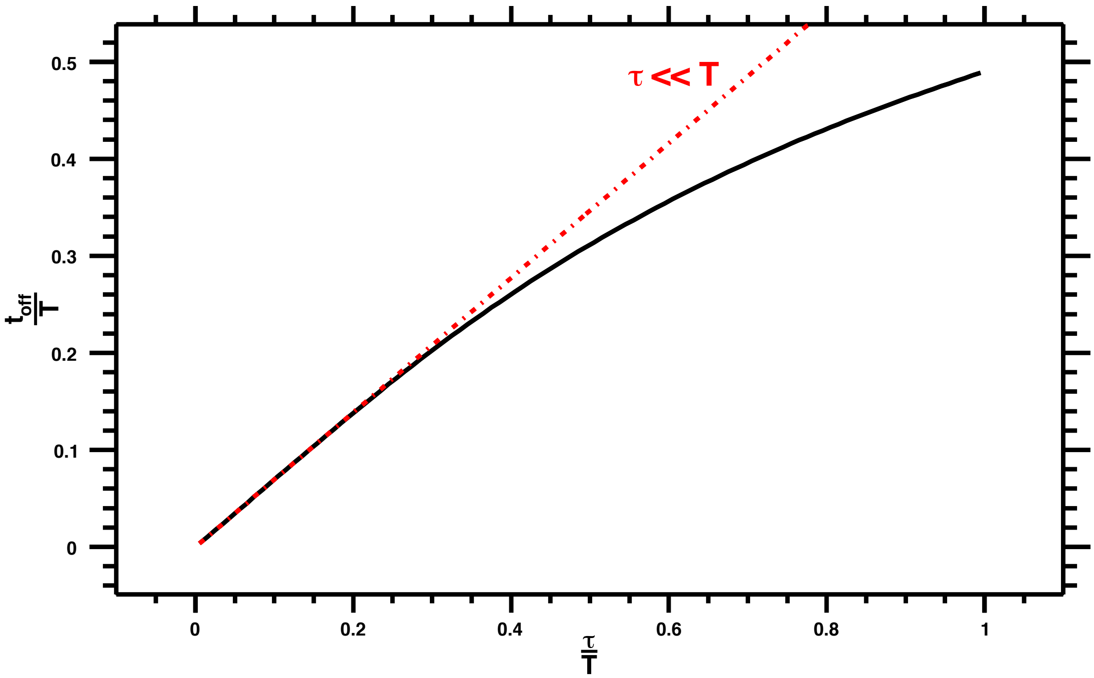

Figure 3: The black is the full calculation of the relative shift in the peak position for a triangular ramp. The red is the $\tau \ll T$ approximation.

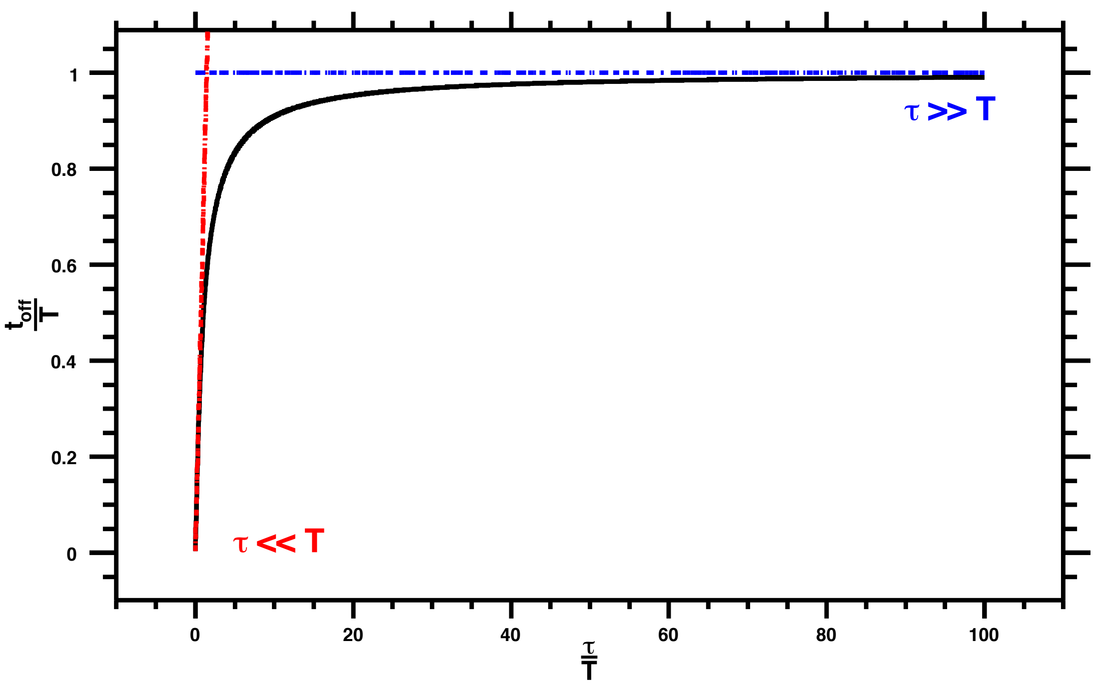

Figure 4: The black is the full calculation of the relative shift in the peak position for a triangular ramp. The red is the $\tau \ll T$ approximation. The blue is the $\tau \gg T$ asymptote.