Solution to atom number dependant rate equation - Jaideep Singh - 2013/02/04

We can write the rate of change of the number of atoms N in the excited state in the following way:

$$\frac{dN}{dt} = -\frac{N}{\tau} - \beta N^2 \tag{1}$$

where τ is the lifetime of the excited state & independant of atom number, β is the rate constant that is linearly proportional to the atom number, and we've assumed only two loss mechanisms. This can be solved by inserting the trial function $N = f(t)\exp(-t/\tau)$, which, after some algebra, gives:

$$\frac{df}{dt} = -\beta f^2 \exp(-t/\tau) \tag{2}$$

This is solved by a separation of variables to give:

$$f(t) = \left[c - \tau\beta \exp(-t/\tau)\right]^{-1} \tag{3}$$

where c is a constant set by the initial conditions. Putting this altogether and rexpressing c in terms of the number of atoms initially in the excited state N_0:

$$N(t) = \frac{N_0 \exp(-t/\tau)}{1 + N_0\tau\beta\left[1 - \exp(-t/\tau)\right]} \tag{4}$$

In the limit that the excited state lifetime is small $\tau\beta \ll 1$, we find:

$$N(t) = N_0 \exp(-t/\tau) \qquad (\tau\beta \ll 1) \tag{5}$$

In the limit that the excited state lifetime is long $\tau\beta \gg 1$, we find:

$$N(t) = \frac{N_0}{1 + N_0\beta t} \qquad (\tau\beta \gg 1) \tag{6}$$