Chapter 5
Symmetries and Conservation Laws

Mackenzie Smith & Stefanie Adams

Parity Operator $\prod$ - Overview

- Spatial reflection of the system $\vec{r} \rightarrow -\vec{r}$
  - e.g. in 3D: $x \rightarrow -x, \quad y \rightarrow -y, \quad z \rightarrow -z$
- Q.M. operators transform according to their (implicit) dependence on $\vec{r}$
  - Odd under parity → $\Pi\hat{A} = -\hat{A}$
  - Even under parity → $\Pi\hat{A} = \hat{A}$
- Useful for solving matrix elements $\langle\phi|\hat{A}|\psi\rangle$
  - If overall parity odd → $\langle\phi|\hat{A}|\psi\rangle = 0$
  - For sph. wave functions → $\Pi|l, m_l, s, m_s\rangle = (-1)^l|l, m_l, s, m_s\rangle$

Parity Operator $\prod$ - Examples

1.) Momentum operator

$$\hat{p} = -i\hbar\vec{\nabla} = -i\hbar\left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right) \Rightarrow \quad \Pi\hat{p} = -\hat{p}$$

odd under parity

x → -x   y → -y   z → -z

2.) Angular momentum operator (pseudo vector)

$$\hat{L} = \hat{r} \times \hat{p} \quad \Rightarrow \quad \Pi\hat{L} = \hat{L}$$

$\vec{r} \to -\vec{r}$ $\vec{p} \to -\vec{p}$

pseudo vectors are even under parity

3.) Electromagnetic fields and potential

- pseudo vector $\vec{B} = \vec{\nabla} \times \vec{A}$ is even under parity, therefore $\vec{A}$ odd under parity
- Electric field $\vec{E} = -\vec{\nabla}\Phi - \partial_t\vec{A}$ is odd under parity, therefore Φ even under parity

Parity Operator $\prod$ - Examples

1.) $\langle \alpha, l = 2, m = 1 | \hat{p}_x | \beta, l = 1, m = 0 \rangle$ → overall even, might be non-zero

even   odd   odd

2.) $\langle \alpha, l = 2, m = 1 | \hat{x}\hat{p}_x | \beta, l = 1, m = 0 \rangle$ → overall odd, matrix element zero

even   odd/odd = even   odd

3.) $\langle \alpha, l = 3, m = 0 | \hat{z}\hat{L}_z | \beta, l = 1, m = 0 \rangle$ → overall odd, matrix element zero

odd   odd/even = odd   odd

Time Reversal Operator Θ - Overview

- Temporal reflection of the system $t \rightarrow -t$
  - e.g. $\vec{r} \rightarrow \vec{r}, \quad \vec{v} \rightarrow -\vec{v}, \quad \vec{E} \rightarrow \vec{E}, \quad \vec{B} \rightarrow -\vec{B}, \quad i\hbar\partial_t \rightarrow i\hbar\partial_t$
- Under time reversal, most operators are even or odd
  - For some operator, B: $\Theta B \Theta^{-1} = \pm B$
- The time reversal operator takes the complex conjugate
  - Thus, it preserves the relation $[J_i, J_j] = i\hbar\varepsilon_{ijk}J_k$
- From this, we can say:
  - $\Theta J \Theta^{-1} = -J$

Time Reversal Problem (Sakurai 4.12)

The Hamiltonian for a spin 1 system is given by:

$$H = AS_z^2 + B\left(S_x^2 - S_y^2\right)$$

Solve this problem *exactly* to find the normalized energy eigenstates and eigenvalues. (A spin-dependent Hamiltonian of this kind actually appears in crystal physics.) Is this Hamiltonian invariant under time reversal? How do the normalized eigenstates you obtained transform under time reversal?

Time Reversal Problem (cont.)

- First, we need to build the S_x, S_y, and S_z matrices

$$S_z \begin{pmatrix}1\\0\\0\end{pmatrix} = \hbar \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$S_z \begin{pmatrix}0\\1\\0\end{pmatrix} = 0$$

$$S_z \begin{pmatrix}0\\0\\1\end{pmatrix} = -\hbar \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$S_z = \hbar \begin{pmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & -1\end{pmatrix}$$

Time Reversal Problem (cont.)

- Then, we build our S_+ and S_- matrices from the following:

$$S_+ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad S_+ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad S_+ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$$

$$S_- \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad S_- \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad S_- \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

Time Reversal Problem (cont.)

- We can then use the relations $S_x = \frac{1}{2}(S_+ + S_-)$ and $S_y = \frac{1}{2i}(S_+ - S_-)$

$$S_+ = \hbar\sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$S_- = \hbar\sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$S_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

Time Reversal Problem (cont.)

- We can now represent our Hamiltonian in matrix form:

$$H = AS_z^2 + B\left(S_x^2 - S_y^2\right)$$

$$H = A\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{B}{2\hbar^2}\left(\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}\right) = \hbar^2 \begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ B & 0 & A \end{pmatrix}$$

Time Reversal Problem (cont.)

- For eigenvalues and eigenvectors, we use det(H-λI) = 0 to find:

$$\lambda_1 = A + B \qquad \lambda_2 = 0 \qquad \lambda_3 = A - B$$

$$|\lambda_1 >= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix} \qquad |\lambda_2 >= \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad |\lambda_3 >= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

Time Reversal Problem (cont.)

- Now, is the Hamiltonian invariant under time reversal?

$$\Theta H \Theta^{-1} = A\Theta S_z^2\Theta^{-1} + B(\Theta S_x^2\Theta^{-1} + \Theta S_y\Theta^{-1})$$

$$= A\Theta S_z\Theta^{-1}\Theta S_z\Theta^{-1} + B\left(\Theta S_x\Theta^{-1}\Theta S_x\Theta^{-1} + \Theta S_y\Theta^{-1}\Theta S_y\Theta^{-1}\right)$$

$$= A\left(-S_z\right)\left(-S_z\right) + B\left(\left(-S_x\right)\left(-S_x\right) + \left(-S_y\right)\left(-S_y\right)\right)$$

$$= AS_z^2 + B\left(S_x^2 + S_y^2\right)$$

$$\Theta H \Theta^{-1} = H$$

Time Reversal Problem (cont.)

- Are the eigenvectors invariant? First, rewrite them in the S_z basis

$$|j=1, m=1>= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|\lambda_1>= \frac{1}{2}(|j=1, m=1> +|j=1, m=-1>)$$

$$|j=1, m=0>= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$|\lambda_2>= |j=1, m=0>$$

$$|j=1, m=-1>= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$|\lambda_3>= \frac{1}{2}(|j=1, m=1> -|j=1, m=-1>)$$

Time Reversal Problem (cont.)

- From Sakurai, for a spin-1 system, $\Theta|j,m>=(-1)^m|j,-m>$

$$\Theta|\lambda_1>=\tfrac{1}{2}(\Theta|1,1>+\Theta|1,-1>)=\tfrac{1}{2}(-|1,-1>-|1,1>)$$

$$\Theta|\lambda_1>=-|\lambda_1>$$

$$\Theta|\lambda_2>=\Theta|1,0>=|1,0>$$

$$\Theta|\lambda_2>=|\lambda_2>$$

$$\Theta|\lambda_3>=\tfrac{1}{2}(\Theta|1,1>-\Theta|1,-1>)=\tfrac{1}{2}(-|1,-1>+|1,1>)$$

$$\Theta|\lambda_3>=|\lambda_3>$$