Chapter 10. Advanced Topics in Angular Momentum

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1 Introduction

This chapter discusses isospin, combining three angular momenta, conventions for Clebsch-Gordon coefficients and Wigner 3j or 6j symbols, irreducible tensor operators, Wigner D matrices and Wigner-Eckart Theorem.

1.1 Previous Exam Problems

Problem 2 (Midterm, Spring 2022), Problem 3 (Subject Exam, Spring 2021), Problem 2 (Midterm, Spring 2021), Problem 2 (Subject Exam, August 2020), Problem 2 (Subject Exam, Spring 2020), Problem 2 (Midterm, Spring 2020), Problem 2 (Practice Spring Midterm).

2 Irreducible Tensor Operators

$$1 = T_0^0$$

$$x = \frac{1}{\sqrt{2}}(T_{-1}^1 - T_1^1)$$

$$y = \frac{i}{\sqrt{2}}(T_{-1}^1 + T_1^1)$$

$$z = T_0^1$$

$$x^2 = \frac{1}{2}\frac{2}{\sqrt{3}}(T_2^2 + T_{-2}^2) - \frac{1}{3}T_0^2 + \frac{1}{3}T_0^0 r^2$$

$$y^2 = -\frac{1}{2}\frac{2}{\sqrt{3}}(T_2^2 + T_{-2}^2) - \frac{1}{3}T_0^2 + \frac{1}{3}T_0^0 r^2$$

$$z^2 = \frac{2}{3}T_0^2 + \frac{1}{3}T_0^0 r^2$$

$$xy = \frac{i}{\sqrt{6}}(T_{-2}^2 - T_2^2)$$

$$xz = \frac{1}{\sqrt{3}}(T^2_{-1} - T^2_1)$$

$$yz = \frac{i}{\sqrt{3}}(T^2_{-1} + T^2_1)$$

3 Wigner-Eckart Theorem

$$\langle \alpha, J, M | T^k_q | \beta, l, m \rangle \sim C^{kq,lm}_{JM}$$

It means:

$$(1) q + m = M$$

$$(2) |k - l| \leq J \leq |k + l|$$

4 Problem (From Subject Exam Spring 2021)

An external electric field, $E_0\hat{x}$, interacts with a system via the perturbative interaction

$$V = -q\vec{E} \cdot \vec{r}$$

(1) (10 pts) Circle the matrix elements which might be non-zero.

$$\langle \alpha, J = 2, M_J = 1 | V | \beta, J = 2, M_J = 0 \rangle$$

$$\langle \alpha, J = 2, M_J = 1 | V | \beta, J = 1, M_J = 0 \rangle$$

$$\langle \alpha, J = 2, M_J = 1 | V | \beta, J = 0, M_J = 0 \rangle$$

(2) (10 pts) Bob and Carol perform a complicated integral to find the matrix element

$$I = \langle \alpha, J = 1, M_J = 0 | z | \beta, J = 0, M = 0 \rangle$$

Ted and Alice wish to calculate

$$J = \langle \alpha, J = 1, M_J = 1 | x | \beta, J = 0, M = 0 \rangle$$

Express $\boldsymbol{J}$ in terms of $\boldsymbol{I}$ and Clebsch-Gordan coefficients. You need not evaluate the Clebsh-Gordan coefficients.

4.1 Solution

Here, it is easy to make mistakes thinking the dot product of two vectors is a scalar and thus the perturbative interaction should be proportional to T_0^0. However, it is important to remember that $\vec{E}$ is external and as a result, we only care for the dependence on $\vec{r}$. And since the electric field is in the $\hat{x}$ direction, we can imagine the interaction only depends on x. Remember,

$$x = \frac{1}{\sqrt{2}}(T_{-1}^1 - T_1^1)$$

Using points (1) and (2) of the Wigner-Eckart Theorem in the last section, it is then clear that only the third option must be zero. So, the right answer to this question will be the first two options.

4.2 Solution

The first step would be to re-write the matrix elements $\boldsymbol{J}$ and $\boldsymbol{I}$ in terms of the irreducible tensor operators.

$$I = \langle \alpha, J = 1, M_J = 0 | T_0^1 | \beta, J = 0, M = 0 \rangle$$

$$J = \langle \alpha, J = 1, M_J = 1 | \frac{1}{\sqrt{2}}(T_{-1}^1 - T_1^1) | \beta, J = 0, M = 0 \rangle$$

We can get rid of the T_{-1}^1 operator from the $\boldsymbol{J}$ matrix element since it will not contribute according to the Wigner-Eckart theorem. So, we can write J as:

$$J = \langle \alpha, J = 1, M_J = 1 | -\frac{1}{\sqrt{2}} T_1^1 | \beta, J = 0, M = 0 \rangle$$

Now, in the final step, again using the Wigner-Eckart Theorem, we can now write $\boldsymbol{J}$ in terms of $\boldsymbol{I}$ as:

$$J = -\frac{1}{\sqrt{2}} \frac{\langle 1, 1 | 1, 1, 0, 0 \rangle}{\langle 1, 0 | 1, 0, 0, 0 \rangle}$$