# Chapter 7 Problem:
# Differential Solid Angle, Structure Function, and Form Factor

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**Relevant Equations Present on Exam Equation Sheet**

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{point}} \tilde{S}(\vec{q}), \tag{1}$$

$$\tilde{S}(\vec{q}) = \frac{1}{N}\left|\sum_{\vec{a}} e^{i\vec{q}\cdot\vec{a}}\right|^2 \tag{2}$$

**Relevant Equations NOT Present on Exam Equation Sheet**

$$\begin{aligned} q &\equiv \vec{k_i} - \vec{k_f} \\ &= k\left[(1-\cos\theta)\,\hat{z} - \sin\theta\cos\phi\;\hat{x} - \sin\theta\sin\phi\;\hat{y}\right], \end{aligned} \tag{3}$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{point}} \left|f\left(\vec{q}\right)\right|^2, \tag{4}$$

$$f(\vec{q}) = \int d\vec{a}\rho\left(\vec{a}\right)e^{i\vec{q}\cdot\vec{a}} \tag{5}$$

**Problem 1** The cross section for scattering a particle with momentum $\vec{k} = k\hat{z}$ off a single target is

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{point}} = \alpha,$$

which is independent of the scattering angle. Here, we assume that the cross section is small. Now, two targets are placed a distance $a$ apart, separated along the $\hat{z}$ axis.

(a) At what scattering angles does the differential cross section, $\frac{d\sigma}{d\Omega}$, equal zero?

(b) At what scattering angles is the differential cross section, $\frac{d\sigma}{d\Omega}$, maximized? What is this maximum value?

**Problem 1 Solution**

We are free to choose the origin wherever it is most convenient. Here we choose the origin between the two targets, so $\vec{a}_1 = -a\hat{z}/2$ and $\vec{a}_2 = a\hat{z}/2$. Using Eqs. 1, 2, and 3,

$$\begin{aligned}\sum_{\vec{a}} e^{i\vec{q}\cdot\vec{a}} &= e^{iq_z a/2} + e^{-iq_z a/2} \\ &= 2\cos\left(\frac{q_z a}{2}\right) \\ \longrightarrow \tilde{S}(\vec{q}) &= 2\cos^2\left(\frac{q_z a}{2}\right) \\ \longrightarrow \frac{d\sigma}{d\Omega} &= 2\alpha\cos^2\left(\frac{q_z a}{2}\right)\end{aligned}$$

Note that $N = 2$, so $\tilde{S}(\vec{q})$ is half of the sum squared.

**(a)**

$\cos x = 0$ when $x = \frac{(2n+1)\pi}{2},\ n \in \mathbb{Z}$, so the differential cross section is zero at

$$\begin{aligned}\frac{q_z a}{2} &= \frac{(2n+1)\pi}{2} \\ \longrightarrow ka(1-\cos\theta) &= (2n+1)\pi \\ \longrightarrow \cos\theta &= 1 - \frac{(2n+1)\pi}{ka}.\end{aligned}$$

So,

$$\cos\theta_{\min} = 1 - \frac{(2n+1)\pi}{ka}.$$

**(b)**

$\cos^2 x = 1$ when $x = n\pi,\ n \in \mathbb{Z}$, so the differential cross section is maximum at

$$\begin{aligned}\frac{q_z a}{2} &= n\pi \\ \longrightarrow ka(1-\cos\theta) &= 2n\pi \\ \longrightarrow \cos\theta &= 1 - \frac{2n\pi}{ka}.\end{aligned}$$

So,

$$\cos\theta_{\max} = 1 - \frac{2n\pi}{ka},$$

with maximum value $d\sigma/d\Omega = 2\alpha$.

**Problem 2** The cross section for scattering a particle with momentum $\vec{k} = k\hat{z}$ off a single target is

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{point}} = \alpha,$$

which is independent of the scattering angle. Here, we assume that the cross section is small. Now, two targets are placed a distance $a$ apart, separated along the $\hat{x}$ axis.

(a) Find the condition for the scattering angles such that the differential cross section, $\frac{d\sigma}{d\Omega}$, equals zero.

(b) At what scattering angles is the differential cross section, $\frac{d\sigma}{d\Omega}$, maximized? What is this maximum value?

**Problem 2 Solution**

We are free to choose the origin wherever it is most convenient. Here we choose the origin between the two targets, so $\vec{a}_1 = -a\hat{x}/2$ and $\vec{a}_2 = a\hat{x}/2$. Using Eqs. 1, 2, and 3,

$$\sum_{\vec{a}} e^{i\vec{q}\cdot\vec{a}} = e^{iq_x a/2} + e^{-iq_x a/2}$$
$$= 2\cos\left(\frac{q_x a}{2}\right)$$
$$\longrightarrow \tilde{S}(\vec{q}) = 2\cos^2\left(\frac{q_x a}{2}\right)$$
$$\longrightarrow \frac{d\sigma}{d\Omega} = 2\alpha\cos^2\left(\frac{q_x a}{2}\right)$$

Note that $N = 2$, so $\tilde{S}(\vec{q})$ is half of the sum squared.

**(a)**

$\cos x = 0$ when $x = \frac{(2n+1)\pi}{2}$, $n \in \mathbb{Z}$, so the differential cross section is zero at

$$\frac{q_x a}{2} = \frac{(2n+1)\pi}{2}$$
$$\longrightarrow -ka\sin\theta\cos\phi = (2n+1)\pi$$
$$\longrightarrow \sin\theta\cos\phi = \frac{(2n+1)\pi}{ka}.$$

So,

$$\sin\theta_{\text{min}}\cos\phi_{\text{min}} = \frac{(2n+1)\pi}{ka}.$$

Note that the minus sign in $q_x$ was dropped since $n \in \mathbb{Z}$ includes positive and negative integers.

**(b)**

$\cos^2 x = 1$ when $x = n\pi$, $n \in \mathbb{Z}$, so the differential cross section is zero at

$$\frac{q_x a}{2} = n\pi$$
$$\longrightarrow -ka\sin\theta\cos\phi = 2n\pi$$
$$\longrightarrow \sin\theta\cos\phi = \frac{2n\pi}{ka}.$$

So,

$$\sin\theta_{\text{max}}\cos\phi_{\text{max}} = \frac{2n\pi}{ka}.$$

with maximum value $d\sigma/d\Omega = 2\alpha$. Note that the minus sign in $q_x$ was dropped since $n \in \mathbb{Z}$ includes positive and negative integers.

**Problem 3** The cross section for scattering a particle with momentum $\vec{k} = k\hat{z}$ off a single point-like target is

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{point}} = \alpha,$$

which is independent of the scattering angle. Here, we assume that the cross section is small. Now, suppose the target is distributed uniformly along a line of length $a$ on the $\hat{z}$ axis.

At what scattering angles does the differential cross section, $\frac{d\sigma}{d\Omega}$, equal zero?

**Problem 3 Solution**

We are free to choose the origin wherever it is most convenient. Here we choose the origin at the midpoint of the distribution. Using Eqs. 3, 4, and 5,

$$
\begin{aligned}
f(\vec{q}) &= \int d\vec{r}\frac{1}{a}e^{i\vec{q}\cdot\vec{r}} \\
&= \int_{-a/2}^{a/2} dr\frac{1}{a}e^{iq_z r} \\
&= \frac{1}{iq_z}\left[e^{iq_z a/2} - e^{-iq_z a/2}\right] \\
&= \frac{2}{q_z}\sin\left(\frac{q_z a}{2}\right)
\end{aligned}
$$

So, the differential cross section is zero when $q_z a = 2n\pi, n \in \mathbb{Z}$. Or,

$$\cos\theta_{\text{min}} = 1 - \frac{2n\pi}{ka}.$$