# QM Subject Exam Preparation (Chapter 3)

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## Background

### Scalar and Vector Potentials

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{A}, \qquad \vec{B} = \vec{\nabla}\times\vec{A}$$

### Schrödinger's Equation

$$i\hbar\partial_t\psi = \left[\frac{1}{2m}(-i\hbar\nabla)^2 + V(r)\right]\psi.$$

Making a minimal substitution of $-i\hbar\nabla \to -i\hbar\nabla - e\vec{A}/c$ and $-i\hbar\partial_t \to -i\hbar\partial_t - e\Phi$, the Schrödinger equation becomes:

$$\frac{(-i\hbar\nabla - e\vec{A}/c)^2}{2m}\psi + V\psi = (i\hbar\partial_t - e\Phi)\psi.$$

The Hamiltonian is then given by

$$H = \frac{1}{2m}(\vec{P} - e\vec{A}/c)^2 + e\Phi$$

### Gauge Invariance

Recall that the Gauge transformation is $\vec{A} \to \vec{A} + \nabla\Lambda(\vec{r}, t)$ and $\Phi \to \Phi - \frac{1}{c}\frac{\partial}{\partial t}\Lambda$ for any function $\Lambda$. In Quantum, we add that $\psi \to \exp\{\frac{1}{\hbar c}ie\Lambda\}\psi$, which is simply adding a phase to the wavefunction. When you combine the changes for $\vec{A}, \vec{\Phi}$, and $\vec{\psi}$ you get invariance.

### Particles in Constant Magnetic Field (Landau Levels)

Consider a particle mass $m$ charge $q$ with $B = B\hat{z}$. Classically, we expect to move following the cyclotron frequency $w_c = \frac{qB}{m}$. Choose $\vec{A} = -By\hat{x}$ for simplicity. The Hamiltonian is then

$$H = \frac{1}{2m}\left(P_x + \frac{q}{c}By\right)^2 + \frac{1}{2m}P_y^2$$

ignoring the motion in the z-direction. Note that $p_x$ is a conserved quantity since $H$ has no explicit $x$ dependence. So we are looking for eigenstates of $P_x$, and we choose

$$\psi(x, y) = \psi(y)e^{ik_x x}$$

where $p_x = \hbar k_x$. Then the Hamiltonian can be rearranged to be in the form of a simple harmonic oscillator:

$$\begin{aligned} H_{k_x} =& \frac{1}{2m}P_y^2 + \frac{1}{2m}\left(\frac{qB}{c}y + \hbar k_x\right)^2 \\ =& \frac{1}{2m}P_y^2 + \frac{m\omega_c^2}{2}\left(y - \left(-\frac{\hbar c k_x}{qB}\right)\right)^2 \end{aligned}$$

Now consider adding a uniform electric field in the $\hat{x}$ direction (assume no initial velocity in the $\hat{z}$-direction). The Hamiltonian looks like:

$$H = \frac{P_x^2}{2m} + \frac{1}{2m}(P_y - eA_y/c)^2 - eEx$$
$$= \frac{P_x^2}{2m} + \frac{e^2B^2}{2mc^2}\left[x - \left(\frac{\hbar ck_y}{eB} + \frac{mc^2E}{eB^2}\right)\right]^2 - \frac{mc^2}{2}\left(\frac{E}{B}\right)^2 - \frac{P_yE}{B}$$
$$= \frac{P_x^2}{2m} + \frac{e^2B^2}{2mc^2}(x - x_0)^2 - \frac{mc^2}{2}\left(\frac{E}{B}\right)^2 - \frac{P_yE}{B}$$

where

$$x_0 = \frac{\hbar ck_y}{eB} + \frac{mc^2E}{eB^2},$$

the center of the harmonic oscillator.

Now suppose we wanted to calculate the average (drift) velocity of the particle in the $\hat{y}$ direction. We can exploit the fact that

$$mv_y = \Pi_y$$

so

$$v_y = \frac{\Pi_y}{m} = \frac{hk_y}{m} - \frac{eBx}{mc}$$

Averaging $v_y$, we substitute in $x_0$ for $x$:

$$\bar{v_y} = \frac{hk_y}{m} - \frac{eBx_0}{mc}$$
$$= \frac{hk_y}{m} - \frac{eB}{mc}\left(\frac{\hbar ck_y}{eB} + \frac{mc^2E}{eB^2}\right)$$
$$= -\frac{Ec}{B}.$$

**Useful Commutation Relations:**

$[r_i, P_j] = 0$ if $i \neq j$

## Practice Final Fall 2019 Question 7

A particle of mass $m$ and charge $e$ is placed in a region with uniform magnetic field $\vec{B}$ along the $\hat{z}$ axis.

a) Write the vector potential that describes the magnetic field such that $\vec{A}$ is in the $\hat{y}$ direction.

b) Write the Hamiltonian with this vector potential.

c) Which quantities commute with the Hamiltonian?

   (i) $P_x$
   (ii) $P_y$
   (iii) $P_z$
   (iv) $P_x - eA_x/c$
   (v) $P_y - eA_y/c$
   (vi) $P_z - eA_z/c$

## Solutions

### Part a

We know that we want $A_\mu$ to only have a term in the $A_y$ term and produce only a magnetic field. In this case, we need to satisfy the following condition (reducing the vector potential to the 3-components as $A_o$=0):

$$\begin{aligned}
\vec{B} &= \vec{\nabla} \times \vec{A} \\
&= \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \times (A_x, A_y, A_z \right) \\
(0, 0, B) &= \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \\
&= \left( 0, 0, \frac{\partial A_y}{\partial x} \right) \\
&= (0, 0, Bx)
\end{aligned}$$

Meaning that $A_y = Bx\hat{y}$.

### Part b

We know that the Hamiltonian has the form of

$$H = \frac{1}{2m}(\vec{P} - e\vec{A}/c)^2 + e\Phi$$

By plugging in the components of $\vec{P}$ and $\vec{A}$ and knowing that $A_o = \Phi = 0$ we get the Hamiltonian to be

$$H = \frac{P_x^2}{2m} + \frac{(P_y - eBx/c)^2}{2m} + \frac{P_z^2}{2m}$$

### Part c

By looking at the Hamiltonian we see that there is an explicit dependence on $x$ which rules out $P_x$. Because $P_x$ is rules out, so is $P_x - eA_x/c$. Recall that $A_y$ has $x$ dependence which we have already ruled out ruling out $P_y - eA_y/c$. Therefore, the only quantities that commute are $P_y$,$P_z$, and $P_z - eA_z/c$ (because $A_z = 0$ so $P_z - eA_z/c = P_z$).