Chapter 2 Review

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Topics Covered in Chapter

Wave Equation

Momentum Space

Charge conservation and Continuity

Potential Problems in 1D

Harmonic Oscillator

Past Problems

Not exactly (it's too easy!), but content shows up mostly in:

- Spherically symmetric bound state problems (in combination with chapter 4)
  - August 2021 problem 4, August 2020 problem 1, Spring 2020 problem 4 part d
- Harmonic oscillators frequently show up in Fermi's golden rule problems
- invart

Bound states

4. Consider a particle of mass $\boldsymbol{m}$ in a three-dimensional potential

$$\boldsymbol{V(r) = -\beta\delta(r-a), \qquad \beta > 0.}$$

(a) (10 pts) In terms of $\boldsymbol{\beta}$ and $\boldsymbol{m}$, what is the minimum value of $\boldsymbol{a}$ for which one has a bound state?

This gives a Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2}\psi(r) + V(r)\psi(r) = E\psi(r)$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2}\psi(r) - \beta\delta(r-a)\psi(r) = E\psi(r)$$

Bound states

$$\begin{aligned}\frac{\partial^2}{\partial r^2}u(r\neq a) &= -\frac{2mE}{\hbar^2}u(r\neq a)\\ &= \frac{2m|E|}{\hbar^2}u(r\neq a)\end{aligned}$$

This is positive because, for a bound state, the energy is negative. This implies the wavefunctions are exponentials. We can also tell that k is the same on both sides.

$$\begin{aligned}-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2}\psi(r) - \beta\delta(r-a)\psi(r) = E\psi(r)\\ \frac{\partial^2}{\partial r^2}\psi(r) = -\frac{2m}{\hbar^2}\left(E+\beta\delta(r-a)\right)\psi(r)\end{aligned}$$

Integrating this over the interval $[a-\varepsilon, a+\varepsilon]$ $(\varepsilon\to 0)$, we get the boundary condition

$$\left.\frac{\partial\psi}{\partial r}\right|_{r=a^+} - \left.\frac{\partial\psi}{\partial r}\right|_{r=a^-} = -\frac{2m\beta}{\hbar^2}u(a)$$

Bound states

We also know that the wavefunction must be continuous across the boundary. Furthermore,

$$\psi(0) = 0$$
$$\psi(r \to \infty) = 0$$

From these boundary conditions, we can derive

$$\psi(r < a) = A \sinh kr$$
$$\psi(r > a) = Be^{-kr}$$

Bound states

At $r = a$, the wavefunction and its slope give, respectively,

$$A \sinh ka = Be^{-ka}$$

$$-kBe^{-ka} - kA \cosh ka = -\frac{2m\beta}{\hbar^2} A \sinh ka$$

$$-kA \sinh ka - kA \cosh ka = -\frac{2m\beta}{\hbar^2} A \sinh ka$$

$$\sinh ka + \cosh ka = \frac{2m\beta}{\hbar^2} \frac{\sinh ka}{k}$$

Bound states

For the minimum number of states, $k \to 0$ as we want the wavefunction to be as evenly spread throughout the space as possible.

$$\lim_{k \to 0} (\sinh ka + \cosh ka) = \lim_{k \to 0} \frac{2m\beta a}{\hbar^2} \frac{\sinh ka}{ka}$$
$$1 = \frac{2m\beta a}{\hbar^2}$$
$$a = \frac{\hbar^2}{2m\beta}$$

Aug 2020 Problem 1

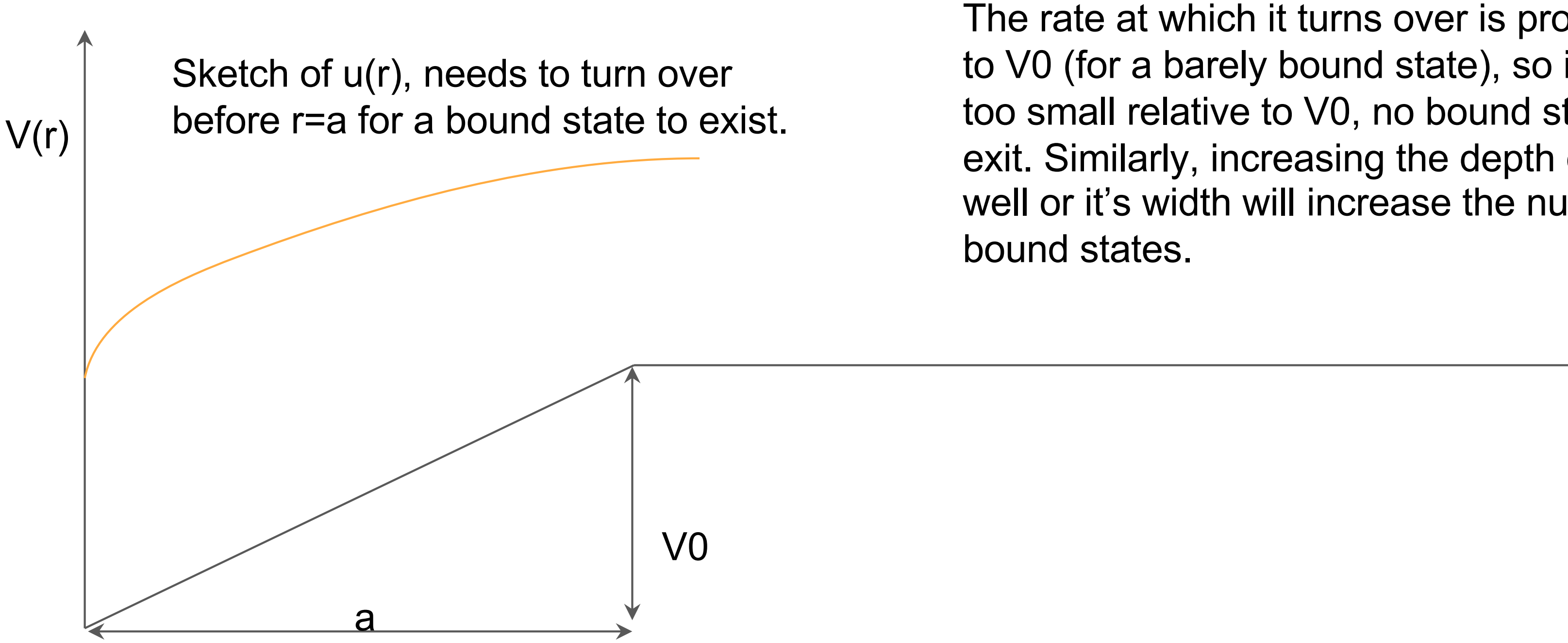
1. (10 pts) A particle of mass m interacts with a spherically symmetric attractive potential,

$$V(r) = \begin{cases} -V_0 + \frac{V_0 r}{a}, & r < a \\ 0, & r > a \end{cases}$$

Circle the true statements below:

- Fixing a, and making V_0 very small, but non-zero, there will always be at least one bound state.
- Fixing V_0, and making a very small, but non-zero, there will always be at least one bound state.
- Fixing a, as the magnitude of V_0 increases, the number of bound states will increase.
- Fixing V_0, as the magnitude of a increases, the number of bound states will increase.

August 2020 Problem 1: Solution



The rate at which it turns over is proportional to V0 (for a barely bound state), so if “a” is too small relative to V0, no bound state will exit. Similarly, increasing the depth of the well or it’s width will increase the number of bound states.

Harmonic Oscillator Overlap

Fermi's golden rule problems sometimes require calculating matrix elements such as $<1|x|0>$ where these are the ground and first excited states of the harmonic oscillator. In these cases, you can avoid integrating by using $x = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^{\dagger}\right)$
Then:

$$< 1|x|0 > =< 1|\sqrt{\frac{\hbar}{2m\omega}}\left(a + a^{\dagger}\right)|0 >$$
$$= \sqrt{\frac{\hbar}{2m\omega}} < 1|a^{\dagger}|0 >= \sqrt{\frac{\hbar}{2m\omega}}$$