Wigner-Eckart Theorem

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A system of $\boldsymbol{N}$ spinless particles are self-bound due to an attractive radially symmetric interaction. States are labeled $|\boldsymbol{\alpha, L, M}\rangle$, where $\boldsymbol{L}$ and $\boldsymbol{M}$ reference the total angular momentum and $\boldsymbol{\alpha}$ accounts for all other quantum labels.

(i) (10 pts) For the matrix elements listed below, circle the NON-ZERO elements: ($\boldsymbol{J, R, P}$ are current, position and momentum operators)

- $\langle \alpha', L' = 2, M' = 1 |\, P_x^2 + P_y^2 \,| \alpha, L = 4, M = 3\rangle$
- $\langle \alpha', L' = 2, M' = 1 |\, P_x P_y \,| \alpha, L = 4, M = 1\rangle$
- $\langle \alpha', L' = 2, M' = 1 |\, \epsilon_{ijk} J_i R_j P_k \,| \alpha, L = 2, M = 1\rangle$
- $\langle \alpha', L' = 2, M' = 1 |\, P_x \,| \alpha, L = 4, M = 3\rangle$

(ii) (10 pts) With great effort, you calculated the matrix element

$$\mathcal{M} = \langle \alpha', L' = 2, M' = 0 |\, P_x^2 + P_y^2 - 2P_z^2 \,| \alpha, L = 4, M = 0\rangle$$

by performing a long and difficult integral. If you were to use the Wigner-Eckart theorem, circle the matrix elements below you could express in terms of $\mathcal{M}$ and Clebsch-Gordan coefficients without having to perform a new integral.

- $\langle \alpha', L' = 2, M' = 2 |\, P_x^2 + P_y^2 - 2P_z^2 \,| \alpha, L = 4, M = 2\rangle$
- $\langle \alpha', L' = 2, M' = 2 |\, P_x^2 + P_y^2 \,| \alpha, L = 4, M = 2\rangle$
- $\langle \alpha', L' = 2, M' = 0 |\, P_x P_y \,| \alpha, L = 4, M = 2\rangle$
- $\langle \alpha', L' = 2, M' = 0 |\, P_x^2 + P_y^2 - 2P_z^2 \,| \alpha, L = 2, M = 0\rangle$

Irreducible Tensor Operators

The irreducible tensor operators are defined as

$$T_q^k \propto r^k Y_{k,q}. \tag{1}$$

where $\boldsymbol{k}$ is the rank of the tensor and $\boldsymbol{q}$ is its projection.
Using this equation you can express polynomial operators as linear combinations of the irreducible tensor operators as they transform similarly under rotations. For example, the position operator $\boldsymbol{x}$ can be written in spherical coordinates as

$$x = r\sin\theta\cos\phi = \frac{1}{2}r\sin\theta(e^{i\phi} + e^{-i\phi}) \propto r(Y_{1,-1} - Y_{1,1}). \tag{2}$$

Thus $\boldsymbol{x}$ transforms as $\boldsymbol{(T^1_{-1} - T^1_1)}$. The set of operators $\boldsymbol{xy}$ can be written as

$$\begin{aligned} xy &= r^2\sin^2\theta\cos\phi\sin\phi \\ &= \frac{1}{4i}r^2\sin^2\theta(e^{i\phi} + e^{-i\phi})(e^{i\phi} - e^{-i\phi}) \\ &= \frac{1}{4i}r^2\sin^2\theta(e^{2i\phi} + e^{-2i\phi}) \\ &\propto r^2(Y_{2,2} + Y_{2,-1}) \end{aligned} \tag{3}$$

which transforms as $\boldsymbol{(T^2_2 + T^2_{-2})}$. The operator $\boldsymbol{x^2 + y^2}$ can written as

$$\begin{aligned} x^2 + y^2 &= r^2\sin^2\theta(\cos^2\phi + \sin^2\phi) \\ &= r^2\sin^2\theta \\ &= r^2(1 - \cos^2\theta) \\ &= \frac{1}{3}r^2(2 - (3\cos^2\theta - 1)) \\ &\propto r^2\Big(\frac{2}{3}Y_{0,0} - \frac{1}{3}Y_{2,0}\Big) \end{aligned} \tag{4}$$

which transforms as $\boldsymbol{\frac{2}{3}r^2T^0_0 - \frac{1}{3}T^2_0}$. The last relevant operator is $\boldsymbol{x^2 + y^2 - 2z^2}$, which can be expressed as

$$\begin{aligned} x^2 + y^2 - 2z^2 &= r^2\Big(\sin^2\theta(\cos^2\phi + \sin^2\phi) - 2\cos^2\theta\Big) \\ &= r^2\Big(\sin^2\theta - 2\cos^2\theta\Big) \\ &= r^2\Big(1 - \cos^2\theta - 2\cos^2\theta\Big) \\ &= -r^2\Big(3\cos^2\theta - 1\Big) \\ &\propto r^2Y_{2,0} \end{aligned} \tag{5}$$

which transforms as $\boldsymbol{T^2_0}$.
Operators that transform similarly under rotation to $\boldsymbol{x, y,}$ and $\boldsymbol{z}$ ($\boldsymbol{P_i, L_i, J_i}$, etc.) would transform as the same irreducible tensor operators $\boldsymbol{T^k_q}$.

Wigner-Eckart Theorem

The general expression of the Wigner-Eckart theorem is

$$\langle \alpha', L', M'|\, T^k_q \,|\alpha, L, M\rangle = \langle k, q, L', M'|L, M\rangle \, \frac{\langle \alpha', L' \| T^{(k)} \| \alpha, L\rangle}{\sqrt{2L'+1}} \tag{6}$$

The double bars in the reduced matrix operator indicate the independence of the matrix element on the projection M. The usefulness of this theorem is that if we calculate the matrix element $\mathcal{M}$ for one combination of M', M and q and generate the reduced matrix element, then we can use it to express all other matrix elements with the same values of L', L and k in terms of the reduced matrix element and the Clebsch-Gordan coefficient.
The selection rules for the matrix element not to be zero are

$$\Delta M = M' - M = q \qquad \text{and} \qquad |L - k| \le L' \le L + k \tag{7}$$

With all these tools now in our toolbox we can proceed with answering the questions.

Solution

(i) We can use the selection rules in (7) to determine which matrix elements are zero.

- $\langle \alpha', L' = 2, M' = 1|\, P_x^2 + P_y^2 \,|\alpha, L = 4, M = 3\rangle$
  As shown in (4) the operator $P_x^2 + P_y^2$ transforms as $\frac{2}{3} r^2 T^0_0 - \frac{1}{3} T^2_0$. The selection rules require that the non zero matrix elements would correspond to the transition between states with
  $$\Delta M = q = 0 \qquad \text{and} \qquad \Delta L = k = 2 \text{ or } 0$$
  which is not the case for this matrix element, since we can see that
  $$M' - M = -2 \qquad \text{and} \qquad |L' - L| = 2$$
- $\langle \alpha', L' = 2, M' = 1|\, P_x P_y \,|\alpha, L = 4, M = 1\rangle$
  As shown in (3), the operator $P_x P_y$ transforms as $T^2_2 + T^2_{-2}$. Thus, this matrix element is zero because $\Delta M = 0 \neq \pm 2$
- $\langle \alpha', L' = 2, M' = 1|\, \epsilon_{ijk} J_i R_j P_k \,|\alpha, L = 2, M = 1\rangle$
  In this case, we can see that this operator is of rank 0. This can be seen by noticing that all three indices, i,j and k are bound (they appear more than once). Since there are no free indices, then this is a rank zero operator, hence a scalar. From our set of irreducible tensor operators, the one which is scalar is T^0_0, which transforms as $Y_{0,0} = \frac{1}{\sqrt{4\pi}}$. Since $\Delta L = \Delta M = 0$ then this is a non zero matrix element.
- $\langle \alpha', L' = 2, M' = 1|\, P_x \,|\alpha, L = 4, M = 3\rangle$
  As shown in (2), the operator P_x transforms as $T^1_1 + T^1_{-1}$. Thus, this matrix element is zero because $\Delta M = -2 \neq \pm 1$.

(ii) $\boldsymbol{\mathcal{M} = \langle \alpha', L' = 2, M' = 0 | \, P_x^2 + P_y^2 - 2P_z^2 \, | \alpha, L = 4, M = 0 \rangle}$

As shown in (5), $\boldsymbol{P_x^2 + P_y^2 - 2P_z^2}$ transforms as $\boldsymbol{T_0^2}$. According to the Wigner-Eckart Theorem (6), we know we can express all matrix elements with $\boldsymbol{L' = 2, L = 4, k = 2}$ in terms of $\boldsymbol{\mathcal{M}}$ and the Clebsch-Gordan coefficients.

- $\boldsymbol{\langle \alpha', L' = 2, M' = 2 | \, P_x^2 + P_y^2 - 2P_z^2 \, | \alpha, L = 4, M = 2 \rangle}$
  Again, as shown in (5), $\boldsymbol{P_x^2 + P_y^2 - 2P_z^2}$ transforms as $\boldsymbol{T_0^2}$ and this is a non zero matrix element, because $\boldsymbol{\Delta M = 0 = q}$. Since $\boldsymbol{L' = 2, L = 4}$, and $\boldsymbol{k = 2}$ then we can express this in terms of $\boldsymbol{\mathcal{M}}$ and the Clebsch-Gordan coefficients.
- $\boldsymbol{\langle \alpha', L' = 2, M' = 2 | \, P_x^2 + P_y^2 \, | \alpha, L = 4, M = 2 \rangle}$
  As shown in (4) this operator transforms as a linear combination of $\boldsymbol{T_0^0}$ and $\boldsymbol{T_0^2}$. This is a non zero matrix element, because $\boldsymbol{\Delta M = 0 = q}$, and again, since $\boldsymbol{L' = 2, L = 4}$, and $\boldsymbol{k = 2}$ then we can express this in terms of $\boldsymbol{\mathcal{M}}$ and the Clebsch-Gordan coefficients.
- $\boldsymbol{\langle \alpha', L' = 2, M' = 0 | \, P_x P_y \, | \alpha, L = 4, M = 2 \rangle}$
  In this case, as shown in (3) the operator transforms as $\boldsymbol{T_{\pm 2}^2}$ and the matrix is non zero because $\boldsymbol{\Delta M = -2 = q}$ from the $\boldsymbol{T_{-2}^2}$ operator. Again, since $\boldsymbol{L' = 2, L = 4}$, and $\boldsymbol{k = 2}$ then we can express this matrix element in terms of $\boldsymbol{\mathcal{M}}$ and the Clebsch-Gordan coefficients. However in this case, since it is not the same irreducible operator T_0^2 in both $\boldsymbol{\mathcal{M}}$ and our matrix element, then in our calculation we should take into account the ratio of the constant factors in front of each operator $\boldsymbol{T_q^k}$, because in this case they will not cancel out.
- $\boldsymbol{\langle \alpha', L' = 2, M' = 0 | \, P_x^2 + P_y^2 - 2P_z^2 \, | \alpha, L = 2, M = 0 \rangle}$
  In this case we can see that $\boldsymbol{L' = 2, L = 2}$, and $\boldsymbol{k = 0}$. Hence we can't express this from our matrix element $\boldsymbol{\mathcal{M}}$.