Chapter 11 QM Review

April 21, 2021

1 Fermi Gases

Example 1.1: – Altered problem 4 from August 2020 subject exam

Consider a ONE-DIMENSIONAL world with protons and neutrons of mass M and massless electrons and neutrinos. Neutrinos and anti-neutrinos can readily exit or enter the system. The system is confined to a large box of length L, has zero net electric charge, had has baryon density of n_B. The protons and neutrons move non-relativistically. The interactions,

$$p + e \leftrightarrow n + \nu, \qquad n \leftrightarrow p + e + \bar{\nu}$$

take place until the energy is minimized.
For each question below, give your answer in terms of n_B, M, $\hbar$, L, and the Fermi wave numbers k_p, k_e and k_n for protons, electrons and neutrons respectively
a) Write three equations expressing the fact that the system is electrically neutral, has fixed net baryon density, and has minimum energy.
b) Solve for k_p, k_e and k_n
Solution
a) The three equations we must solve for in terms of Fermi wave numbers and the constants given in the problem are

$$n_B = n_n + n_p, \qquad n_e = n_p, \qquad \epsilon_f^n = \epsilon_f^p + \epsilon_f^e$$

First we solve n_n, n_p, and n_e in terms of the Fermi wave numbers. The density is a function of the Fermi momentum,

$$\begin{aligned} N &= (2s+1)\frac{L}{2\pi\hbar}\int^{p_f} dp \\ n &= \frac{N}{L} = \frac{2s+1}{2\pi\hbar}p_f \\ n &= \frac{k_f}{\pi} \end{aligned} \tag{1.1}$$

Thus we have the three densities

$$n_n = \frac{k_n}{\pi}, \qquad n_p = \frac{k_p}{\pi}, \qquad n_e = \frac{k_e}{\pi}$$

which we can use to solve the three equations mentioned above.

$$\begin{aligned} n_B &= n_n + n_p \\ &= \frac{1}{\pi}(k_n + k_p) \end{aligned} \tag{1.2}$$

$$\begin{aligned} n_p &= n_e \\ k_p &= k_e \end{aligned} \tag{1.3}$$

$$\begin{aligned} \epsilon_f^n &= \epsilon_f^p + \epsilon_f^e \\ \frac{\hbar^2 k_n^2}{2M} &= \frac{\hbar^2 k_p^2}{2M} + \hbar k_e c \end{aligned} \tag{1.4}$$

b) Using the three equations we can solve for k_p, k_e and k_n.

$$\begin{aligned} \frac{\hbar^2 k_n^2}{2M} &= \frac{\hbar^2 k_p^2}{2M} + \hbar k_e c \\ \frac{2M}{\hbar} k_e c + k_p^2 - k_n^2 &= 0 \\ \frac{2Mc}{\hbar} k_p + k_p^2 - (\pi n_B - k_p)^2 &= 0 \\ \frac{2Mc}{\hbar} k_p + k_p^2 - k_p^2 - \pi^2 n_B^2 + 2k_p \pi n_B &= 0 \\ (\frac{2Mc}{\hbar} + 2\pi n_B) k_p &= \pi^2 n_B^2 \end{aligned} \tag{1.5}$$

$$\begin{aligned} k_p &= \frac{\pi^2 n_B}{\frac{2Mc}{\hbar} + 2\pi n_B} \\ k_e &= k_p \\ k_n &= \pi n_B - k_p \end{aligned} \tag{1.6}$$

2 Correlations in a Fermi Gas

Q: What is a correlation function?

A: If the states are not correlated, then it would just be the product of both states. A correlation function mean there's a relation between the two function.

Q: What does the following function represent?

$$\rho(\vec{x}) = \sum_s \Psi_s^\dagger(\vec{x})\Psi_s(\vec{x})$$

A: First, just assume we have one spin. We then get $\Psi_s^\dagger(\vec{x})\Psi_s(\vec{x})$. This is something, that when we put this agaist a state, we will get the the value of that state at a position x. Looking at the example of just one state, it would be,

$$
\begin{aligned}
&= \langle\alpha| \Psi_s^\dagger(\vec{x})\Psi_s(\vec{x}) |\alpha\rangle && (2.1)\\
&= \langle 0| \alpha^\dagger\Psi_s^\dagger(\vec{x})\Psi_s(\vec{x})\alpha |0\rangle \qquad \text{using } \{\Psi(\vec{x}), a^\dagger\} = \phi(\vec{x}) && (2.2)\\
&= \langle 0| (\phi^*(\vec{x}) - \Psi^\dagger(\vec{x})\alpha)(\alpha^\dagger\Psi(\vec{x}) - \phi(\vec{x})) |0\rangle \qquad \text{using } \Psi(\vec{x})|0\rangle = 0 && (2.3)\\
&= \phi^*(x)\phi(x) && (2.4)
\end{aligned}
$$

Now you know how this works for one state, try adding spin and multiple states.

Q: What does the following represent?

$$\langle\alpha,\beta,\gamma,\ldots| \Psi_s^\dagger(\vec{x})\Psi_s(\vec{x}) |\alpha,\beta,\gamma,\ldots\rangle$$

A: Because the states are the same on both sides, just imagine $\Psi_s^\dagger(\vec{x})\Psi_s(\vec{x})$ going through each bra-ket pair and creating a wave function. We get the sum $\sum_{k\in\alpha.\beta,\gamma,\ldots} \phi_k^*(\vec{x})\phi_k(\vec{x})$. This is the wave function of all the single-particle states. Given a position, it gives the density of each state at position $\vec{x}$.

Q: What happens when one label in the bra and ket differs?

A: This represents a transition probability between the two state. Thus we only get the wave function of the corresponding label. This represent the transition probability between two states. The fermi operator selected only the two states that we needed.

$$\langle\alpha,\beta,\gamma',\ldots| \Psi_s^\dagger(\vec{x})\Psi_s(\vec{x}) |\alpha,\beta,\gamma,\ldots\rangle \to \phi_{\gamma'}^*(\vec{x})\phi_\gamma(\vec{x})$$

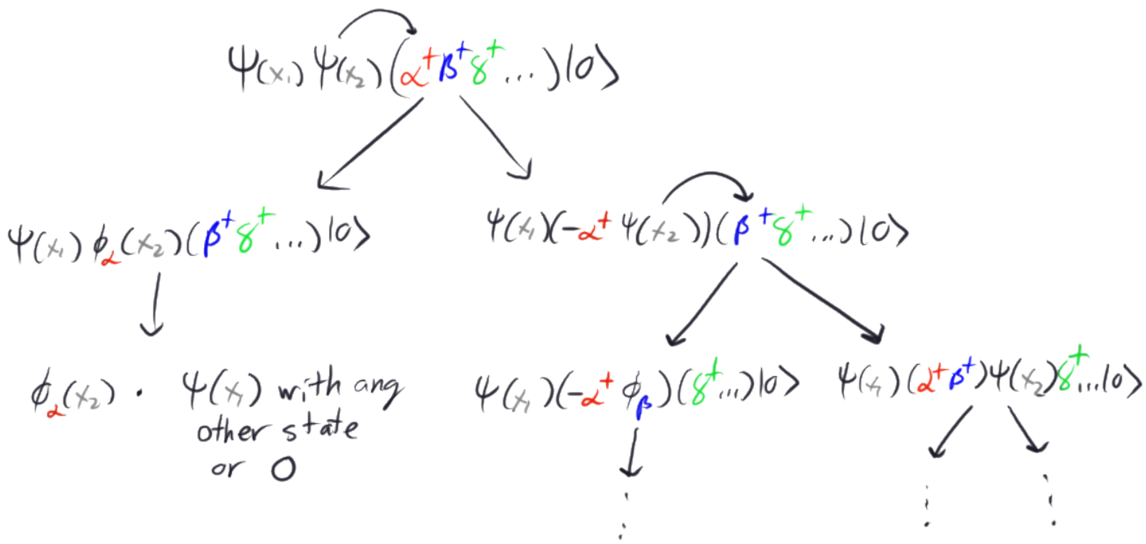
Q: Substituting in the commutation relationship makes the calculation take too long. If we have multiple field and creation operators, what is the best way to think about it?

$$(\Psi(x_1)\Psi(x_2)\ldots)(\alpha^\dagger\beta^\dagger\gamma^\dagger\ldots)|0\rangle$$

$$\{\Psi(\vec{x}), a^\dagger\} = \phi(\vec{x})$$

$$\Psi(\vec{x})|0\rangle = 0$$

A: $\Psi(x_i)$ walks through each creation operator and when it pass through one, it has two choices. It can either walk through and pick up a negative sign, or it could destroy it and turn into the result of the anti-commutator, in this case, the wave function. This effectively gives all the combinations of wave functions at each position for each state. If there's more field operators than particles, we would get zero. The next question is what happen when we have more creation operators than field operators?



Q: What does the following represent?

$$\rho_2(x_1, x_2) = \langle\phi|\, \Psi^\dagger(x_1)\Psi^\dagger(x_2)\Psi(x_2)\Psi(x_1)\, |\phi\rangle$$

$$|\phi\rangle = \prod_{k<k_f} a_k^\dagger\, |0\rangle$$

A: Let's first look at the ket. It is all the particles in a Fermi Gas with momentum k less than k_f, the Fermi momentum. It's a one dimensional gas in a region of length L and ρ_2 would give us the probability of having two particles close to each other. In the final result, we should see ρ_2 go to zero as $r = |x_1 - x_2| \to 0$.

Example 2.2: – Correlations in a Small Fermi Gas
Here, we will calculate the correlation in a small Fermi Gas of just two states. $k < k_f = k_0, k_1$. Consider this in one-dimension in a large region of length L.

$$\Psi(x) = \sum_{\vec{k}} a_{\vec{k}} \frac{e^{ikx}}{\sqrt{L}}$$

$$\rho_2(x_1, x_2) = \langle\phi|\, \Psi^\dagger(x_1)\Psi^\dagger(x_2)\Psi(x_2)\Psi(x_1)\, |\phi\rangle$$

$$|\phi\rangle = a_0^\dagger a_1^\dagger\, |0\rangle$$

1. This problem will be done in detail so you can believe that the general solution in Scott notes, Example 11.5, works. First we will write out the right side of the correlation function.

$$\Psi(x_2)\Psi(x_1)a_0^\dagger a_1^\dagger |0\rangle = \tag{2.5}$$

$$= [\sum_{k_1,k_2<k_f} \frac{e^{ik_2x_2}}{\sqrt{L}} \frac{e^{ik_1x_1}}{\sqrt{L}} a_{k_2}a_{k_1}](a_0^\dagger a_1^\dagger) |0\rangle \tag{2.6}$$

$$= [\frac{e^{ik_0x_2}e^{ik_0x_1}}{L} a_0a_0 + \frac{e^{ik_0x_2}e^{ik_1x_1}}{L} a_0a_1$$
$$+\frac{e^{ik_1x_2}e^{ik_0x_1}}{L} a_1a_0 + \frac{e^{ik_1x_2}e^{ik_1x_1}}{L} a_1a_1](a_0^\dagger a_1^\dagger) |0\rangle$$

- Note: We can think of this as $a_{k2}a_{k1}$ coupling to the e's

- Image the $a's$ as walking through the $a^\dagger{}'s$. Only those which matches will live. If it's in opposite order, it switch and we get a negative sign.

$$= [\frac{e^{ik_1x_2}e^{ik_0x_1}}{L} - \frac{e^{ik_0x_2}e^{ik_1x_1}}{L}] |0\rangle$$

2. Now solve for the left side.

$$\langle 0| a_0a_1[\sum_{k_1,k_2<k_f} \frac{e^{-ik_2x_2}}{\sqrt{L}} \frac{e^{-ik_1x_1}}{\sqrt{L}} a_{k_2}^\dagger a_{k_1}^\dagger]$$

$$= \langle 0| [\frac{e^{-ik_0x_1}e^{-ik_1x_2}}{L} - \frac{e^{-ik_1x_1}e^{-ik_0x_2}}{L}]$$

3. Multiplying this two side together, we match terms with the same x_1 and x_2. Terms with the same k's will cancel while terms with different k's won't.

$$\langle \phi| \Psi^\dagger(x_1)\Psi^\dagger(x_2)\Psi(x_2)\Psi(x_1) |\phi\rangle = \frac{1}{L^2}+\frac{1}{L^2}-\frac{e^{ix_1(k_1-k_0)}e^{ix_2(k_0-k_1)}}{L^2}-\frac{e^{ix_1(k_0-k_1)}e^{ix_2(k_1-k_0)}}{L^2}$$

$$= \frac{2}{L^2} - \frac{e^{i(k_1-k_0)(x_1-x_2)}}{L^2} - \frac{e^{-i(k_1-k_0)(x_1-x_2)}}{L^2}$$
$$= \frac{2}{L^2} - \frac{2\cos((k_1-k_0)(x_1-x_2))}{L^2}$$

$$\rho_2(r \equiv x_1 - x_2) = \frac{2}{L^2} - \frac{2\cos((k_1-k_0)r)}{L^2}$$

And here we see that $\rho_2(0) = 0$.