Problem 13.4. Consider the Dirac representation,

$$\beta = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} \qquad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$$

and the chiral representation,

$$\beta = \begin{pmatrix} 0 & -\mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} \qquad \vec{\alpha} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix}$$

The spinors, $u_\uparrow$ and $u_\downarrow$, represent positve-energy eigenvalues of the Dirac equation assuming the momentum is along the z axis.

$$(m\beta + p_z\alpha_z)u(p_z) = Eu(p_z)$$

The spin labels, $\uparrow$ and $\downarrow$ refer to the positive and negative values of the spin operator,

$$\Sigma_z = \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}$$

Write the four-component spinors $u_\uparrow$ and $u_\downarrow$ in terms of p, E and m:

(a) in the Dirac representation.

(b) in the chiral representation.

(c) in the limit $p_z \to 0$ for both representations.

(d) in the limit $p_z \to \infty$ for both representations.

(a) The generalized, unnormalized spin up and spin down eigenvectors of the spin operator will be,

$$u_\uparrow = \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix} \qquad u_\downarrow = \begin{pmatrix} 0 \\ a \\ 0 \\ b \end{pmatrix}$$

where a and b are complex, and can include phase factors.
Every solution of the Dirac equation has the condition that

$$E^2 = p^2 + m^2 \tag{1}$$

In the Dirac representation, the eigenvalue problem for $u_\uparrow$ becomes

$$\begin{pmatrix} m & 0 & p & 0 \\ 0 & m & 0 & -p \\ p & 0 & -m & 0 \\ 0 & -p & 0 & -m \end{pmatrix} \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix} = E \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix}$$

This leads to the system of equations

$$ma + pb = Ea$$
$$pa - mb = Eb$$

$a = (pb)/(E - m)$, which means

$$u_\uparrow \propto \begin{pmatrix} (pb)/(E-m) \\ 0 \\ b \\ 0 \end{pmatrix} \propto \begin{pmatrix} p \\ 0 \\ E - m \\ 0 \end{pmatrix}$$

Normalizing this such that $u_\uparrow^* u_\uparrow = 1$...

$$\begin{aligned} p^2 + (E-m)^2 &= C^2 \\ p^2 + E^2 - 2mE + m^2 &= C^2 \\ E^2 - m^2 + E^2 - 2mE + m^2 &= C^2 \\ 2E^2 - 2mE &= C^2 \end{aligned}$$

leads to the solution

$$u_\uparrow = \frac{1}{\sqrt{2E(E-m)}} \begin{pmatrix} p \\ 0 \\ E - m \\ 0 \end{pmatrix}$$

Following the same steps for $u_\downarrow$ leads to the following system of equations...

$$
\begin{aligned}
ma - pb &= Ea \\
-pa - mb &= Eb
\end{aligned}
$$

which will lead to an unnormalized solution of

$$
u_{\downarrow} \propto \begin{pmatrix} 0 \\ -pb/(E-m) \\ 0 \\ b \end{pmatrix} \propto \begin{pmatrix} 0 \\ -p \\ 0 \\ E-m \end{pmatrix}
$$

which will lead to the normalized solution of

$$
u_{\downarrow} = \frac{1}{\sqrt{2E(E-m)}} \begin{pmatrix} 0 \\ -p \\ 0 \\ E-m \end{pmatrix}
$$

(b) In the chiral representation, the eigenvalue problem for $u_{\uparrow}$ becomes

$$
\begin{pmatrix} p & 0 & -m & 0 \\ 0 & -p & 0 & -m \\ -m & 0 & -p & 0 \\ 0 & -m & 0 & p \end{pmatrix} \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix} = E \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix}
$$

This leads to the system of equations

$$
\begin{aligned}
pa - mb &= Ea \\
-ma - pb &= Eb
\end{aligned}
$$

$a = (-mb)/(E-p)$, which means

$$
u_{\uparrow} \propto \begin{pmatrix} (-mb)/(E-p) \\ 0 \\ b \\ 0 \end{pmatrix} \propto \begin{pmatrix} -m \\ 0 \\ E-p \\ 0 \end{pmatrix}
$$

Normalizing this such that $u_{\uparrow}^{*} u_{\uparrow} = 1$...

$$
\begin{aligned}
m^2 + (E-p)^2 &= C^2 \\
m^2 + E^2 - 2Ep + p^2 &= C^2 \\
E^2 - p^2 + E^2 - 2Ep + p^2 &= C^2 \\
2E^2 - 2Ep &= C^2
\end{aligned}
$$

leads to the solution

$$
u_\uparrow = \frac{1}{\sqrt{2E(E-p)}} \begin{pmatrix} -m \\ 0 \\ E-p \\ 0 \end{pmatrix}
$$

Following the same steps for $u_\downarrow$ leads to the following system of equations...

$$
\begin{aligned}
-pa - mb &= Ea \\
-ma + pb &= Eb
\end{aligned}
$$

which will lead to an unnormalized solution of

$$
u_\downarrow \propto \begin{pmatrix} 0 \\ (-mb)/(E+p) \\ 0 \\ b \end{pmatrix} \propto \begin{pmatrix} 0 \\ -m \\ 0 \\ E+p \end{pmatrix}
$$

which will lead to the normalized solution of

$$
u_\downarrow = \frac{1}{\sqrt{2E(E+p)}} \begin{pmatrix} 0 \\ -m \\ 0 \\ E+p \end{pmatrix}
$$

(c) In the low momentum limit, the Dirac representation for $u_\uparrow$ has the following system of equations...

$$
\begin{aligned}
ma &= Ea \\
-mb &= Eb
\end{aligned}
$$

and $u_\downarrow$ has the following system of equations...

$$\begin{aligned} ma &= Ea \\ -mb &= Eb \end{aligned}$$

which have the solutions

$$u_\uparrow = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad u_\downarrow = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

In the chiral representation, $u_\uparrow$ has the following system of equations...

$$\begin{aligned} -mb &= Ea \\ -ma &= Eb \end{aligned}$$

and $u_\downarrow$ has the following system of equations...

$$\begin{aligned} -mb &= Ea \\ -ma &= Eb \end{aligned}$$

which have the solutions

$$u_\uparrow = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \qquad u_\downarrow = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

(d) In the high momentum limit, the Dirac representation for $u_\uparrow$ has the following system of equations...

$$\begin{aligned} pb &= Ea \\ pa &= Eb \end{aligned}$$

and $u_\downarrow$ has the following system of equations...

$$\begin{aligned} -pb &= Ea \\ -pa &= Eb \end{aligned}$$

which have the solutions

$$u_\uparrow = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad u_\downarrow = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

In the chiral representation, $u_\uparrow$ has the following system of equations...

$$\begin{aligned} pa &= Ea \\ -pb &= Eb \end{aligned}$$

and $u_\downarrow$ has the following system of equations...

$$\begin{aligned} -pa &= Ea \\ pb &= Eb \end{aligned}$$

which have the solutions

$$u_\uparrow = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad u_\downarrow = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Problem 14.2. Consider the coherent state defined by

$$|\eta\rangle = e^{-\eta^*\eta/2} e^{\eta a^\dagger} |0\rangle \, .$$

(a) Show that η can also be written as

$$|\eta\rangle = e^{-\eta^* a + \eta a^\dagger} |0\rangle$$

(b) Show that the overlap of two states is given by

$$\langle \eta' | \eta \rangle = e^{-|\eta'|^2/2 - |\eta|^2/2 + \eta'^* \eta}.$$

Solution: (a) Let

$$|\chi\rangle = e^{-\eta^* a + \eta a^\dagger} |0\rangle \, .$$

Use the Baker-Campbell-Hausdorff lemma,

$$e^{A+B} = e^A e^B e^{-C/2}, \;\; C = [A, B],$$

and the commutation relation

$$[-\eta^* a, \eta a^\dagger] = -\eta^* \eta [a, a^\dagger] = -\eta^* \eta$$

to show that

$$\begin{aligned} |\chi\rangle &= e^{-\eta^* a} e^{\eta a^\dagger} e^{\eta^* \eta/2} |0\rangle \\ &= e^{\eta^* \eta} e^{-\eta^* a} |\eta\rangle \\ &= e^{\eta^* \eta} \sum_n \frac{(-\eta^* a)^n}{n!} |\eta\rangle \, . \end{aligned}$$

Then use that the coherent state $|\eta\rangle$ is an eigenstate of the destruction operator with eigenvalue η,

$$\begin{aligned} |\chi\rangle &= e^{\eta^* \eta} \sum_n \frac{(-\eta^* \eta)^n}{n!} |\eta\rangle \\ &= e^{\eta^* \eta} e^{-\eta^* \eta} |\eta\rangle \\ &= |\eta\rangle \, . \end{aligned}$$

(b) Using the definition of $|\eta'\rangle$,

$$\begin{aligned} \langle \eta' | \eta \rangle &= \langle 0 | \, e^{-|\eta'|^2/2} e^{\eta'^* a} |\eta\rangle \\ &= e^{-|\eta'|^2/2} \langle 0 | \sum_n \frac{(\eta'^* a)^n}{n!} |\eta\rangle \, . \end{aligned}$$

As in part (a), use $a |\eta\rangle = \eta |\eta\rangle$ to show

$$\langle \eta' | \eta \rangle = e^{-|\eta'|^2/2} e^{\eta'^* \eta} \langle 0 | \eta \rangle \, .$$

Now do the same using the definition of $|\eta\rangle$,

$$\begin{aligned} \langle \eta' | \eta \rangle &= e^{-|\eta'|^2/2} e^{\eta'^* \eta} \langle 0 | \, e^{-|\eta|^2/2} e^{\eta a^\dagger} |0\rangle \\ &= \exp\left(\frac{-|\eta'|^2}{2} + \eta'^* \eta - \frac{|\eta|^2}{2} \right) \langle 0 | \sum_n \frac{(\eta a^\dagger)^n}{n!} |0\rangle \\ &= \exp\left(\frac{-|\eta'|^2}{2} + \eta'^* \eta - \frac{|\eta|^2}{2} \right), \end{aligned}$$

noting that only the $n = 0$ term in the sum survives.