Chapter 3, 12 Presentation

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1 Introduction

Chapters 3 and 12 deal with adjusting quantum mechanics to accommodate classical electromagnetism and methods for studying many electron systems.

Relevant equations: The magnetic vector potential satisfies the following relations

$$\vec{B} = \nabla \times \vec{A}$$

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

In 4-Vector form, it is given as:

$$A^{\alpha} = \left(\frac{\phi}{c}, \vec{A}\right)$$

In a convenient form, the fields satisfy the following Lorentz Transformation:

$$\vec{E_{\parallel}}' = \vec{E_{\parallel}}$$
$$\vec{B_{\parallel}}' = \vec{B_{\parallel}}$$
$$\vec{E_{\perp}}' = \gamma\left(\vec{E_{\perp}} + \vec{v} \times \vec{B}\right)$$
$$\vec{B_{\perp}}' = \gamma\left(\vec{B_{\perp}} - \frac{1}{c^2}\vec{v} \times \vec{E}\right)$$

2 Subject Exam Spring 2000, Problem 4

An electron is placed in a constant magnetic field of strength B which lies along the z axis. The electron also experiences an electric field E which lies along the y axis. Neglect the coupling of the spin to the magnetic field.

1. (5 pts) Write down a vector potential $\mathbf{A}(\mathbf{r}, t)$ with $\mathbf{A}$ being solely along the y axis that results in the electromagnetic field described above.

   In order for the scalar potential to vanish, we need to find $\vec{A}$ s.t.

$$\vec{B} = B_0\hat{z} = \nabla \times \vec{A}$$

$$\vec{E} = E_0\hat{y} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

To solve, we apply the formula for the cross product, simplifying according to the problem statement:

$$\begin{aligned}
\nabla \times \vec{A} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}} \\
&= \frac{\partial A_y}{\partial x}\hat{\mathbf{z}} \\
&= B_0 \hat{z}
\end{aligned}$$

Solving the simple system of DEs for $\vec{A}$ we obtain

$$\vec{A} = (B_0 x - cE_0 t)\,\hat{y}$$

2. (5 pts) Write the Hamiltonian for an electron in the field described above. From chapter 3, the Hamiltonian is given by

$$\begin{aligned}
H &= \frac{1}{2m}\left[\mathcal{P}^2 - \frac{e}{c}(\vec{\mathcal{P}}\cdot\vec{A} + \vec{A}\cdot\vec{\mathcal{P}}) + \left(\frac{e}{c}\right)^2 A^2\right] \\
&= \frac{1}{2m}\left[P_x^2 + P_y^2 + P_z^2 - \frac{e}{c}\vec{A}\cdot\vec{\mathcal{P}} + \left(\frac{e}{c}\right)^2 A^2\right] \\
&= \frac{1}{2m}\left[P_x^2 + P_y^2 + P_z^2 - \frac{e}{c}A_y P_y + \left(\frac{e}{c}A_y\right)^2\right] \\
&= \frac{1}{2m}\left[P_x^2 + P_z^2 + \left(P_y - \frac{e}{c}A_y\right)^2\right]
\end{aligned}$$

We desire to write this in the form of the QHO in order to make solving the 3rd part easier.

$$\begin{aligned}
H &= \frac{1}{2m}\left[\mathcal{P}^2 - \frac{e}{c}(\vec{\mathcal{P}}\cdot\vec{A} + \vec{A}\cdot\vec{\mathcal{P}}) + \left(\frac{e}{c}\right)^2 A^2\right] \\
&= \frac{1}{2m}\left[P_x^2 + P_z^2 + \left(P_y - \frac{e}{c}A_y\right)^2\right] \\
&= \frac{1}{2m}\left[P_x^2 + P_z^2 + \left(P_y - \frac{e}{c}B_0 x + \frac{e}{c}cE_0 t\right)^2\right] \\
&= \frac{1}{2m}\left[P_x^2 + P_z^2 + m^2\omega^2\left(x - x_0 - v_0 t\right)^2\right]
\end{aligned}$$

With the following definitions:

$$\begin{aligned}
m\omega &= \frac{e}{c}B_0 \\
v_0 &= \frac{cE_0}{B_0} \\
P_y &= x_0 m\omega = \frac{eB_0 x_0}{c}
\end{aligned}$$

3. (5 pts) Assuming the wave function is of the form

$$\psi(\mathbf{r}, t) = e^{ik_y y + ik_z z}\phi_{k_y,k_z}(x, t)$$

write the wave equation for $\phi_{k_y,k_z}(x,t)$ where $\hbar k_y$ and $\hbar k_z$ are the eigenvalues of P_y and P_z.

From the form above, we can see that $\phi_{k_y,k_z}(x,t)$ is up to a phase and time evolution, the solution to the QHO with argument $x - x_0 - v_0 t$. Thus we can say that

$$\phi_{k_y,k_z}(x,t) = \phi_n(x - x_0 - v_0 t) * e^{\frac{-i\epsilon_n t}{\hbar}}$$

where ϕ_n is the nth excited state of the QHO and ϵ_n is by definition

$$\epsilon_n = \frac{\hbar^2 k_z^2}{2m} + \frac{m v_0^2}{2} + \left(n + \frac{1}{2}\right)\hbar\omega = \frac{\hbar^2 k_z^2}{2m} + \frac{mc^2 E_0}{2B_0} + \left(n + \frac{1}{2}\right)\hbar\frac{eB_0}{c}$$

3 Chapter 12, Exercise 2

1. One electron moves in a one-dimensional system and feels the interaction of two atoms. Approximate the interaction between the electrons and the atoms with the potential

$$V(x - R) = -\beta\delta(x - R)$$

where R is the position of an atom.

Assuming that the atoms are a distance r apart, we can see that our wavefunctions are

$$\begin{aligned}
\psi_I(x) &= Ae^{k(x+r/2)} \\
\psi_{II}(x) &\sim e^{kx} + e^{-kx} \sim \cosh kx \\
\psi_{III}(x) &= Ae^{-k(x-r/2)}
\end{aligned}$$

Applying the boundary condition that ψ is continuous and lumping all our constants together in one clump, we see that

$$A = \cosh\frac{kr}{2}.$$

Given the delta potential, our second boundary condition

$$\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\psi(x)|_{y+\epsilon} - \frac{\partial}{\partial x}\psi(x)|_{y-\epsilon}\right) = -\beta\psi(y)$$

yields us

$$\begin{aligned}
\frac{\hbar^2}{2m}\left(-Ak + k\sinh\frac{-kr}{2}\right) &= -\beta A \\
-Ak + k\sinh\frac{-kr}{2} &= \frac{-2m\beta A}{\hbar^2} \\
-k\sinh\frac{kr}{2} &= A\left(k - \frac{2m\beta}{\hbar^2}\right) \\
k\sinh\frac{kr}{2} &= \cosh\frac{kr}{2}\left(\frac{2m\beta}{\hbar^2} - k\right) \\
k\tanh\frac{kr}{2} &= \frac{2m\beta}{\hbar^2} - k \\
\tanh\frac{kr}{2} &= \frac{2m\beta}{\hbar^2 k} - 1
\end{aligned}$$

2. Find the potential between the two atoms at small r,

$$V(r \to 0) \sim V(r = 0) - \alpha r$$

that is, find $V(r = 0)$ and α. Do this by expanding the transcendental equation in terms of r. Hint: First, find $V(r = 0)$ by solving the transcendental equation with $r = 0$. Take derivatives of the transcendental equation with respect to r, then solve for dk/dr at $r = 0$, and finally find dE/dr to obtain α.

Evaluating our answer in part 1 at $r = 0$, we get

$$\begin{aligned}
0 &= \frac{2m\beta}{\hbar^2 k} - 1 \\
k &= \frac{2m\beta}{\hbar^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
E &= -\frac{\hbar^2 k^2}{2m} \\
&= -\frac{\hbar^2}{2m}\left(\frac{2m\beta}{\hbar^2}\right)^2 \\
&= -\frac{2m\beta^2}{\hbar^2}.
\end{aligned}$$

Following the hint and taking the derivative of the transcendental equation with respect to r, we get

$$\begin{aligned}
\frac{d}{dr}\tanh\frac{kr}{2} &= \frac{d}{dr}\left(\frac{2m\beta}{\hbar^2 k} - 1\right) \\
\frac{k}{2}\operatorname{sech}^2\frac{kr}{2}|_{r=0} &= -\frac{2m\beta}{\hbar^2 k^2}\frac{dk}{dr} \\
\frac{k}{2} &= -\frac{2m\beta}{\hbar^2 k^2}\frac{dk}{dr} \\
-\frac{\hbar^2 k^3}{4m\beta} &= \frac{dk}{dr} \\
-\frac{\hbar^2}{4m\beta}\left(\frac{2m\beta}{\hbar^2}\right)^3 &= \frac{dk}{dr} \\
-\frac{2m^2\beta^2}{\hbar^4} &= \frac{dk}{dr}|_{r=0}.
\end{aligned}$$

Taking $dE/dr = \alpha$, we get

$$\begin{aligned}
\alpha &= \frac{dE}{dr} \\
&= \frac{dE}{dk}\frac{dk}{dr} \\
&= -\frac{\hbar^2 k}{m}\left(-\frac{2m^2\beta^2}{\hbar^4}\right) \\
&= \frac{\hbar^2}{m}\left(\frac{2m\beta}{\hbar^2}\right)\left(\frac{2m^2\beta^2}{\hbar^4}\right) \\
&= \frac{4m^2\beta^3}{\hbar^4}.
\end{aligned}$$

In the end, we find

$$\begin{aligned}
V(r\to 0) &\sim V(r=0) - \alpha r \\
&= -\frac{2m\beta^2}{\hbar^2} - \frac{4m^2\beta^3}{\hbar^4} r.
\end{aligned}$$

3. Find the potential between the two atoms at large r,

$$V(r\to\infty) = -\gamma\exp\left(-2k_\infty r\right)$$

that is, find γ. Hint: Use first order perturbation theory, assuming the unperturbed wave function is the bound state of one well, and the perturbation is the interaction with the second well.

Our wavefunction is described as

$$\psi = Ae^{-k|r|}$$

where A, our normalization constant, is found to be $\sqrt{k}$. Using first-order perturbation, we find

$$\begin{aligned} V_{\text{new}} &= \int dr\, \psi^* V_{\text{old}} \psi \\ &= \int dr\, ke^{-2k|r|}(-\beta)\delta(r - r_o) \\ &= -\beta k e^{-2kr_o}, \end{aligned}$$

thus, $\gamma = k\beta$ where $k = 2m\beta/\hbar^2$.