Quantum Subject Exam Review

Serikow, Z.; Wantz, A.

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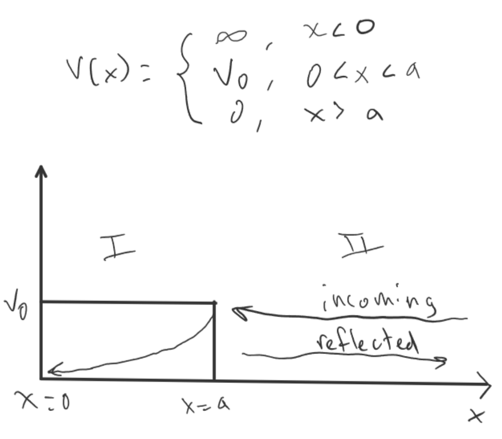
1 Potential Step

Consider a particle of mass m approaching a positive potential step from the $x = +\infty$ direction with wave number k. The potential step is larger than the kinetic energy of the approaching particle. For $x > a$, the wave function has the form

$$\psi(x) = e^{-ikx} - e^{2i\delta(k)} e^{ikx} \tag{1.1}$$



1.1 a)

Find the Phase shift $\delta(k)$:
First, we note that we have the usual boundary conditions: (1) ψ is continuous at $x = a$, (2) first derivative is continuous at x=a, and (3) $\psi = 0$ at $x = 0$. In region I, the wavefunction will decay, so we have exponential solutions:

$$\psi_I = Ae^{-qx} + Be^{qx}, \quad q = \sqrt{2m(V_0 - E)/\hbar^2} \tag{1.2}$$

The coefficients A and B must be equal and opposite for $\psi(0) = 0$, so that

$$\psi_I = Ae^{-qx} - Ae^{qx} = A \sinh(qx) \tag{1.3}$$

Applying BC (1), we find

$$\psi_I(a) = \psi_{II}(a) \tag{1.4}$$

$$A \sinh(qa) = e^{-ika} - e^{2i\delta(k)} e^{ika} = e^{i\delta(k)}(e^{-ika} e^{-i\delta(k)} - e^{i\delta(k)} e^{ika}) \tag{1.5}$$

where $k = \sqrt{(2mE/\hbar^2)}$. Applying BC (2),

$$qA \cosh(qa) = e^{i\delta(k)}(-ike^{-ika} e^{-i\delta(k)} - ike^{i\delta(k)} e^{ika}) \tag{1.6}$$

Dividing the two boundary conditions,

$$\frac{\psi_I(a)}{\partial_x \psi_I(a)} = \frac{\psi_{II}(a)}{\partial_x \psi_{II}(a)} \tag{1.7}$$

$$\frac{\tanh(qa)}{q} = \frac{\tan(ka + \delta)}{k} \tag{1.8}$$

Finally solving for δ, we find

$$\boxed{\delta = \arctan\left(\frac{k \tanh(qa)}{q}\right) - ka}$$

1.2 b)

What is $\delta(k)$ in the limit that $V_0 \longrightarrow \infty$?
As $V_0 \longrightarrow \infty$, $q \longrightarrow \infty$, since $q = \sqrt{2m(V_0 - E)/\hbar^2}$. Thus,

$$\lim_{q \to \infty} \tanh(qa) = \frac{e^{qa} - e^{-qa}}{e^{qa} + e^{-qa}} \approx \frac{e^{qa}}{e^{qa}} = 1 \tag{1.9}$$

Thus we have

$$\arctan \frac{k}{q} \longrightarrow \arctan \frac{k}{\infty} \longrightarrow 0 \tag{1.10}$$

$$\therefore \boxed{\delta = -ka \text{ for } V_0 \longrightarrow \infty} \tag{1.11}$$

2 Delta Function Well

A particle of mass m feels the following potential

$$V(x) = -\beta\delta(x), \quad \beta > 0. \tag{2.1}$$

Find the binding energy of the ground state. How many bound states are there?

2.1 a)

Since we are looking for the binding energy, the energy of the particle must be less than zero, $E < 0$. We must find a relation for the energy, which is done by solving the Schrödinger equation and applying the appropriate boundary conditions. The Schrödinger equation gives

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} - \beta\delta(x)\psi = E\psi. \tag{2.2}$$

We divide the problem into two regions: one with $x < 0$ and the other with $x > 0$. In both these regions, the delta function vanishes and the Schrödinger equation reduces to:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi. \tag{2.3}$$

$$\frac{d^2\psi}{dx^2} = \kappa^2\psi, \quad \kappa = \frac{\sqrt{-2mE}}{\hbar} \tag{2.4}$$

The general solution is simply a linear combination of exponentials

$$\psi(x) = \begin{cases} Ae^{-\kappa x} + Be^{\kappa x}, & \text{x} < 0 \\ Ce^{-\kappa x} + De^{\kappa x}, & \text{x} > 0. \end{cases} \tag{2.5}$$

We now apply boundary conditions. Because the wave function cannot blow up as $x \longrightarrow \infty$ and $x \longrightarrow -\infty$, we find that the solutions in both regions only contain one term

$$\psi(x) = \begin{cases} Be^{\kappa x}, & \text{x} < 0 \\ Ce^{-\kappa x}, & \text{x} > 0. \end{cases} \tag{2.6}$$

The wave function must be continuous at $x = 0$. This means that $B = C$. The Schrödinger equation becomes

$$\psi(x) = \begin{cases} Ce^{\kappa x}, & \text{x} \leq 0 \\ Ce^{-\kappa x}, & \text{x} \geq 0. \end{cases} \tag{2.7}$$

Finally, we apply the condition regarding the first derivative of the wave function. Normally, the first derivative must be continuous. This condition is not true at points where the potential is infinite, such as at $x = 0$ for the delta function. If we integrate the Schrödinger equation as written in eq. 2.2 from $-\varepsilon$ to ε and take the limit that $\varepsilon \longrightarrow 0$ we find

$$\left.\frac{d\psi}{dx}\right|_{+\varepsilon} - \left.\frac{d\psi}{dx}\right|_{-\varepsilon} = \frac{2m}{\hbar^2}\lim_{\varepsilon\longrightarrow 0}\int_{-\varepsilon}^{+\varepsilon} \beta\delta(x)\psi(x)dx \tag{2.8}$$

$$\left.\frac{d\psi}{dx}\right|_{+\varepsilon} - \left.\frac{d\psi}{dx}\right|_{-\varepsilon} = -\frac{2m\beta}{\hbar^2}\psi(0). \tag{2.9}$$

Explicitly evaluating eq. 2.9 using eq. 2.7 results in

$$\kappa = \frac{m\beta}{\hbar^2}. \tag{2.10}$$

Inserting κ from eq. 2.4 and rearranging yields the energies for the bound states

$$E = \frac{m\beta^2}{2\hbar^2}. \tag{2.11}$$

We notice that E depends solely on constants and β. Because β is fixed, there is only one bound state of the delta-function well, so this is the energy of the ground state we were looking for.

2.2 b)

Suppose we are asked to find the wave function for this singly bound state of the delta-function well. We have done most of the work already since we have $\psi(x)$ from eq. 2.7 and κ from eq. 2.10. We just need to normalize $\psi(x)$ to find C:

$$\int_{-\infty}^{+\infty} |\psi(x)|^2 dx = C^2 \int_{-\infty}^{0} e^{2\kappa x} dx + C^2 \int_{0}^{+\infty} e^{-2\kappa x} dx = \frac{C^2}{\kappa} = 1 \implies C = \frac{\sqrt{m\beta}}{\hbar}. \tag{2.12}$$

The wave equation is then

$$\psi(x) = \begin{cases} \frac{\sqrt{m\beta}}{\hbar} e^{m\beta x/\hbar^2}, & \text{x} \leq 0 \\ \frac{\sqrt{m\beta}}{\hbar} e^{-m\beta x/\hbar^2}, & \text{x} \geq 0. \end{cases} \tag{2.13}$$