Chapters 6 and 7 : Perturbations, Fermi's Golden Rule, and Scattering

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Problem 1

(Practice Exam Fall 2019 #8)
Consider a *Bryan* particle of mass **m** confined to a one-dimensional potential,

$$\text{V(x)} = \begin{cases} \infty & x \leq -a \\ 0 & -a \leq x \leq a \\ \infty & a \leq x \end{cases}$$

It can decay to a *Brianna* particle of the same mass, but the Brianna particle does not feel the potential. The Hamiltonian matrix element responsible for the decay is

$$\langle 0, Bryan|V|k, Brianna\rangle = \frac{\alpha e^{-k^2b^2/2}}{\sqrt{L}}$$

where the momentum of the Brianna particle is $\hbar\mathbf{k}$, the large length of the plane wave $|k\rangle$ is **L**, and the constant α is small. What is the Bryan-particle decay rate? Present your answer in terms of α , **a**, **b**, **V** and **m**.

Solution

To solve this problem, we begin with Fermi's Golden Rule

$$\frac{\partial}{\partial t}P_{i\to n}(t) = \Gamma_{i\to n} = \frac{2\pi}{\hbar}\sum_k |V_{ni}|^2\delta(E_n - E_i) \tag{1}$$

This important equation relates the matrix element $|V_{ni}|$ to the decay rate Γ. We are able to use this equation by first identifying the matrix element as $< 0, Bryan|V|k, Brianna >$. In this problem, the matrix element is given, so you can easily square it to get in the correct format for the decay rate:

$$|V_{ni}|^2 = | < 0, Bryan|V|k, Brianna > |^2 = \frac{\alpha^2 e^{-k^2b^2}}{L} \tag{2}$$

and substituted it above,

$$\Gamma_{i\to n} = \frac{2\pi}{\hbar}\sum_k \frac{\alpha^2 e^{-k^2b^2}}{L}\delta(E_n - E_i) \tag{3}$$

The dimensional integral over all states can replace the summation of Fermi's Golden Rule:

$$\sum_k \to (\frac{L}{2\pi})^D \int_{-\infty}^{\infty} d^D k \tag{4}$$

where D is dimensions. In this case, we are dealing with a one-dimensional potential. Transitioning to the integral form of Fermi's Golden Rule then gives us the relation:

$$\Gamma_{i\to n} = \frac{\alpha^2}{\hbar}\int_{-\infty}^{\infty} e^{-k^2b^2}\delta(E_n - E_i)dk \tag{5}$$

Now we need to determine the initial and final energies to evaluate the delta function of this decay equation. Since we have a Bryan particle decaying into a Brianna particle the initial energy will be equal to the energy of the Bryan particle, $E_i = E_{Bryan} = E$. The Brianna particle is a plane wave with energy $E_n = E_{Brianna} = \frac{\hbar^2k^2}{2m}$, which is our final energy. Thus the δ function becomes:

$$\delta(E_n - E_i) \to \delta(E - \frac{\hbar^2k^2}{2m}) \tag{6}$$

For this δ function to apply to the integral, we want to isolate the factor of k inside the δ function. To do this, the Stanley Theorem gives us:

$$\int_{-\infty}^{\infty} dkh(k)\delta(f(k) - f_0) = \frac{h(k)}{|g'(k)|} \tag{7}$$

where $g(k) = f(k) - f_0$. In the case of this problem, this applies as:

$$\Gamma_{i\to n} = \int_{-\infty}^{\infty} e^{-k^2b^2}\delta(E_n - E_i)dk = \int_{-\infty}^{\infty} e^{-k^2b^2}\delta(E - \frac{\hbar^2k^2}{2m})dk = \frac{e^{-k^2b^2}}{|-\frac{\hbar^2k}{2m}|} \tag{8}$$

From the delta function, we know:

$$E = \frac{\hbar^2k^2}{2m} \tag{9}$$

$$k = \sqrt{\frac{2mE}{\hbar^2}} \tag{10}$$

Which allows a simplification to the final value for the decay rate:

$$\Gamma_{i\to n} = \frac{e^{-2mEb^2/\hbar^2}}{|-\frac{\hbar^2k}{2m}|} = \sqrt{\frac{2m}{\hbar^2E}}e^{-2mEb^2/\hbar^2} \tag{11}$$

Problem 2

(Subject Exam Spring 2020 #3)
Consider a beam of particles of momentum $\hbar\mathbf{k}$ elastically scattering off three identical targets placed at the following positions:

$$\vec{R_1} = (x = 0, y = 0, z = 0) \tag{12}$$

$$\vec{R_2} = (x = R, y = 0, z = 0) \tag{13}$$

$$\vec{R_3} = (x = -R, y = 0, z = 0) \tag{14}$$

The direction of the scattered particles is denoted in spherical coordinates, with θ describing the direction relative to the beam (**z**) axis, and ϕ measuring the direction relative to the **x** axis in the **x-y** plane, i.e. if the wave number for the scattered particle is $\vec{k}^{(f)}$,

$$\vec{k}_z^{(f)} = \vec{k}^{(f)}\cos\theta,\ \vec{k}_x^{(f)} = \vec{k}^{(f)}\sin\theta\cos\phi,\ \vec{k}_y^{(f)} = \vec{k}^{(f)}\sin\theta\sin\phi$$

(a) Consider scattering observed in the **x-z** plane ($\phi = 0$). At what polar angles θ will the differential cross section disappear?
(b) Repeat for scattering observed in the **y-z** plane ($\phi = 90°$).

Solution part a

In beginning to solve this scattering problem it is useful to draw the situation of the three target particles and the incident beam in the coordinate plane.

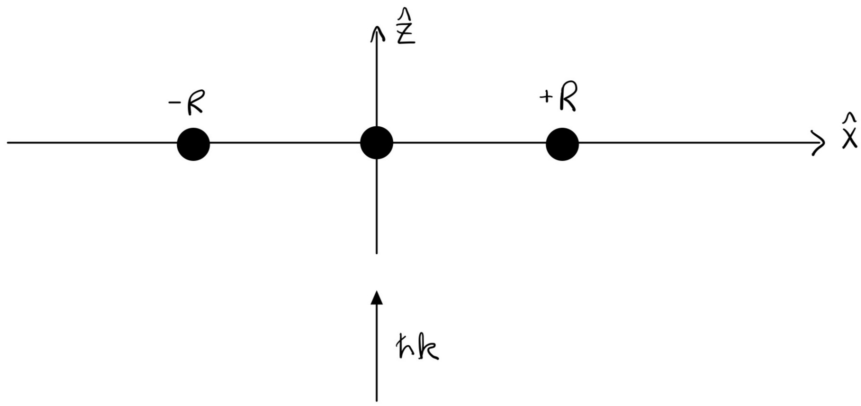

Figure 1: Black dots are identical target particles along the x-axis, ħk is the incident beam along the z-axis.

We then start with the Fourier transform of the structure function that gives the contributions of the individual target particles to the cross section. This is given as:

$$S(\vec{q}) = \sum_{\delta a} e^{i\vec{q}\cdot\delta\vec{a}} \tag{15}$$

Applying the given positions of the target particles in this problem, the structure function yields:

$$S(\vec{q}) = e^{i\vec{q}\cdot\vec{R}_1} + e^{i\vec{q}\cdot\vec{R}_2} + e^{i\vec{q}\cdot\vec{R}_3} \tag{16}$$

In the problem, we are given the components of k with the polar angle θ and the azimuthal angle ϕ. Since the particles are along the x-axis, the values $\delta\vec{a}$ are $\pm R\hat{x}$ and $0\hat{x}$. Thus the structure function becomes:

$$S(\vec{q}) = e^{i\vec{q}\cdot R\hat{x}} + e^{0} + e^{-i\vec{q}\cdot R\hat{x}} \tag{17}$$

and performing the dot product between the momentum transfer and the vector to the particle yields,

$$S(\vec{q}) = e^{ikRsin(\theta)cos(\phi)} + 1 + e^{-ikRsin(\theta)cos(\phi)} \tag{18}$$

Now, since the problem states that for part a, scattering is observed in the x-z plane, $\phi = 0$, so $\cos\phi = 1$, the equation becomes:

$$S(\vec{q}) = 1 + e^{ikRsin(\theta)} + e^{-ikRsin(\theta)} = 1 + 2cos(kRsin(\theta)) \tag{19}$$

The specific aim in this problem of finding the angle θ in which the differential cross section disappears, so we set $S(\vec{q}) = 0$. This allows us to write:

$$S(\vec{q}) = 1 + 2cos(kRsin(\theta)) = 0 \tag{20}$$

and we can re-write it such that

$$cos(kRsin(\theta)) = -\frac{1}{2} \tag{21}$$

Since Cosine of $\frac{2\pi}{3} + 2\pi n$ is $-\frac{1}{2}$, we can now write:

$$\frac{2\pi}{3} + 2\pi n = kRsin(\theta) \tag{22}$$

which solves as:

$$\theta = sin^{-1}(\frac{2\pi}{3kR} + \frac{2\pi n}{kR}) \tag{23}$$

where n is an integer resulting from the periodic nature of the cosine in $2cos(kRsin(\theta))$ and gives us the number of zero scattering angles as a result of the arc-sine being limited to values below 1.

Solution part b

We are now observing scattering in the y-z plane, so now apply $\phi = 90°$ to Eq 18 to get:

$$cos(90°) = 0 \tag{24}$$

and substituting back into the structure function yields:

$$S(\vec{q}) = e^{ikRsin(\theta)*0} + 1 + e^{-ikRsin(\theta)*0} = 3 \tag{25}$$

Since there is no θ value for which this structure function equals 0, the differential cross section will not disappear.