Chapter 8 Review problems

Name: Carissa Myers, Roy Salinas

Fall Final 1998, Problem 5

A spherically symmetric potential has the form

$$V(r) = \frac{\hbar^2}{2m}\beta\delta(r-a) \tag{1}$$

1. As a function of the asymptotic momentum p, find the s-wave phase shift in terms of a, β, and the particle's mass m. Assume $\beta > 0$.
2. What is the cross section in the limit of zero relative momentum?
3. Now, assume $\beta < 0$. Find the minimum magnitude $|\beta|$ necessary for the creation of a bound state.

Part a

We are looking at the asymptotic limit of p, which means we have an unbound wave function whose solution far from the scattering center behaves like a plane wave. We consider the following,

1. Consider whether $E > V$ or $E < V$
   1.1. $E > V$ in this case.
2. Define the appropriate wave functions in the region
   2.1. In $r < a$, the solution must pass through the origin (spherical bessel functions).
   2.2. In $r > a$, the solution contains both the scattered wave and the incoming wave (asymptotic limit of hankel functions is one approach).
3. Apply continuous and smoothly varying boundary conditions.
   3.1. One will have to thread carefully with the smoothly varying boundary condition due to the delta potential.

It is always helpful to get a "mental image" of the problem at hand (Figure 1). Looking at $r < a$ region and remembering that the radial solutions to $u(r)$ are a linear combination of spherical Bessel and Neumman functions,

$$u(r) = rR(r) = Arj_l(kr) + Brn_l(kr) \rightarrow u(r) = Arj_0(kr) = Ar\frac{\sin(kr)}{kr} \qquad r < a \tag{2}$$

We have gotten rid of the Neuman solution as that solution blows up at the origin. In arriving at the $r > a$ wave function, we can use the hankel functions in their asymptotic representation.

$$R_l(r)\Big|_{r>a} = \frac{1}{2}\left(e^{2i\delta} h_l(kr) + h_l^*(kr)\right) \tag{3}$$

$$h_l(kr)\big|_{r>a} \approx \frac{(-i)^{l+1} e^{ikr}}{kr} \tag{4}$$

$$h_l^*(kr)\big|_{r>a} \approx \frac{(i)^{l+1} e^{-ikr}}{kr} \tag{5}$$

$$\Rightarrow R_{l=0}(r)\Big|_{r>a} = \frac{-ie^{2i\delta}e^{ikr} + ie^{-ikr}}{2kr} = \frac{e^{i\delta}(-ie^{i(kr+\delta)} + ie^{-i(kr+\delta)})}{2kr} = \frac{e^{i\delta}\sin(kr+\delta)}{kr} \tag{6}$$

$$u(r) = rR(r) = \frac{e^{i\delta}\sin(kr+\delta)}{k} \qquad r > a \tag{7}$$

Where $h_l(kr)$ is the outgoing wave (the one with the phase) and $h_l^*(kr)$ is the incoming wave. Seeing the phase attached to the outgoing wave should make sense if one thinks about the order of events: the outgoing wave has interacted with the potential and picks up a phase; the incoming wave has not interacted with the potential so there should be no phase. Our wave functions for the incoming beam then are[1]

$$u(r) = \begin{cases} A\dfrac{\sin(kr)}{k}, & r < a \\ \dfrac{e^{i\delta}\sin(kr+\delta)}{k}, & r > a \end{cases} \tag{8}$$

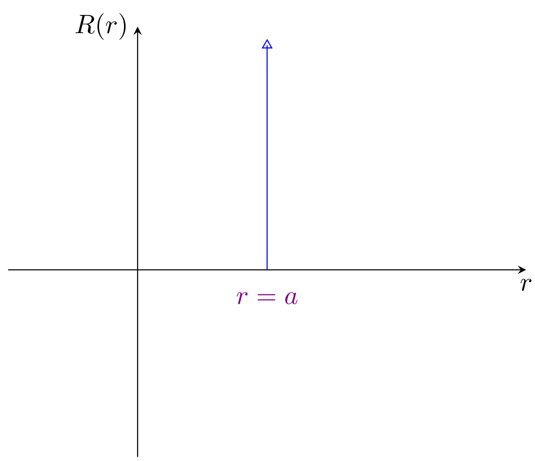

Figure 1: Repulsive delta potential at $r = a$

All that is left to do is apply both boundary conditions and find δ. Applying the continuous BC,

$$u(r=a)_I = u(r=a)_{II} \tag{9}$$

$$A\sin(ka) = e^{i\delta}\sin(ka+\delta) \tag{10}$$

$$\Rightarrow A = \frac{e^{i\delta}\sin(ka+\delta)}{\sin(ka)} \tag{11}$$

Applying the second BC,

$$\partial_r u(r)\Big|_{a+\epsilon} - \partial_r u(r)\Big|_{a-\epsilon} = \frac{-2m}{\hbar^2}\lim_{\epsilon\to 0}\left(\int_{a-\epsilon}^{a+\epsilon}(E - V(r))u(r)dr\right) \tag{12}$$

[1]The factors of k are "unimportant" and can be ignored and one still arrives at the same answer. Look at Spring 2000, Problem 7, part b

When evaluating the integral over the integrand $Eu(r)$, note that as $\epsilon \to 0$, the wave functions look the same on both sides and there is no phase. Thus, integrating $Eu(r)$ over the symmetric interval would go to zero.

$$\begin{aligned}
\lim_{\epsilon\to 0}\left(\partial_r u(r)\Big|_{a+\epsilon} - \partial_r u(r)\Big|_{a-\epsilon}\right) &= \frac{2m}{\hbar^2}\lim_{\epsilon\to 0}\left(\int_{a-\epsilon}^{a+\epsilon}(V(r))u(r)dr\right) &(13)\\
&= \frac{2m}{\hbar^2}\lim_{\epsilon\to 0}\left(\int_{a-\epsilon}^{a+\epsilon}\beta\frac{\hbar^2}{2m}\delta(r-a)u(r)dr\right) &(14)\\
&= \beta u(a) &(15)\\
e^{i\delta}\cos(ka+\delta) - A\cos(ka) &= \beta A\frac{\sin(ka)}{k} &(16)
\end{aligned}$$

Rearranging terms and plugging in for our constant A,

$$\begin{aligned}
e^{i\delta}\cos(ka+\delta) &= A\left(\cos(ka)+\frac{\beta}{k}\sin(ka)\right) &(17)\\
e^{i\delta}\cos(ka+\delta) &= \frac{e^{i\delta}\sin(ka+\delta)}{\sin(ka)}\left(\cos(ka)+\frac{\beta}{k}\sin(ka)\right) &(18)\\
\frac{\sin(ka)}{\left(\cos(ka)+\frac{\beta}{k}\sin(ka)\right)} &= \frac{\sin(ka+\delta)}{\cos(ka+\delta)} = \tan(ka+\delta) &(19)
\end{aligned}$$

Taking the inverse tangent of both sides and subtracting by ka, we arrive at

$$\delta_0 = \tan^{-1}\left(\frac{\sin(ka)}{\cos(ka)+\frac{\beta}{k}\sin(ka)}\right) - ka \tag{20}$$

for a s wave in a spherically symmetric repulsive potential. To check this indeed makes sense, if set $\beta = 0$ (which is saying there is no potential), we expect there to be no phase (the wave does not scatter and continuous traveling onward). Doing so,

$$\begin{aligned}
\delta_0 &= \tan^{-1}\left(\frac{\sin(ka)}{\cos(ka)}\right) - ka &(21)\\
&= \tan^{-1}(\tan(ka)) - ka &(22)\\
&= ka - ka = 0 &(23)
\end{aligned}$$

which is what we'd expect.

Part b

The low-energy limit of the cross section is[2],

$$\sigma = \frac{4\pi}{k^2}\sum_l (2l+1)\sin^2(\delta_l) \tag{24}$$

where this is the total cross-section with contributions from different orbital angular momentum values, l. The problem is asking to find σ as $p \to 0$, which means our incident beam is going to zero. If the incoming beam has low energy, the dominating orbital angular momentum value is $l = 0$[3]. Thus, the scattering cross-section becomes

$$\sigma = \frac{4\pi}{k^2}\sin^2(\delta_0) \tag{25}$$

[2] To see where this comes from, refer to section 8.1 of Scott's notes, Ch. 6.5 of Sakurai (Second Ed.), or Ch 11 (particularly 6.4) of Zettili (Second Ed.).

[3] Look at the "Additional" section for a brief explanation

We begin by taking the limit $p \to 0$ ($k \to 0$) of δ_0

$$\lim_{k\to 0} \delta_0 = \lim_{k\to 0} \left(\tan^{-1} \left(\frac{\sin(ka)}{\cos(ka) + \frac{\beta}{k}\sin(ka)} \right) - ka \right) \tag{26}$$

$$= \lim_{k\to 0} \left(\frac{\sin(ka)}{\cos(ka) + \frac{\beta}{k}\sin(ka)} - ka \right) \tag{27}$$

$$= \lim_{k\to 0} \left(\frac{ka}{1 + \frac{\beta}{k} ka} - ka \right) \tag{28}$$

$$= \lim_{k\to 0} ka \left(\frac{1}{1+\beta a} - 1 \right) \tag{29}$$

$$= \lim_{k\to 0} -\frac{ka^2\beta}{1+\beta a} \tag{30}$$

$$\sigma \approx = \lim_{k\to 0} \frac{4\pi}{k^2} \sin^2 \left(-\frac{ka^2\beta}{1+\beta a} \right) = \frac{4\pi}{k^2} \left(-\frac{ka^2\beta}{1+\beta a} \right)^2 \tag{31}$$

$$= 4\pi a^2 \left(\frac{a\beta}{1+\beta a} \right)^2 \tag{32}$$

Part c

We are asked to find the minimum value of β that results in a bound state ($E < V$). As always, try to get a "mental image" of the problem, and note $\beta < 0$ this time around meaning it is an attractive potential (Figure 2). We begin with the aforementioned procedure:

1. Consider whether $E > V$ or $E < V$
   1.1. In this case, it is the latter since we are looking for a bound state.
2. Define the appropriate wave functions in the region
   2.1. In $r < a$, the solution must decay and pass through the origin.
   2.2. In $r > a$, the solution must decay as $r \to \infty$.
3. Apply continuous and smoothly varying boundary conditions.
   3.1. One will have to thread carefully with the smoothly varying boundary condition due to the delta potential.
4. Since we are looking for the minimum of some quantity, expect to take the limit of something.
   4.1. In this case, we are looking for β_{min}, which is related to the binding energy, or going back a layer, k.

Defining our wave functions,

$$u(r) = \begin{cases} Ae^{kr} + Be^{-kr} = A\sinh(kr), & r < a \\ e^{-k(r-a)}, & r > a \end{cases} \tag{33}$$

where we have imposed the condition that the wave function must pass through the origin for $r < a$ and must decay to zero in $r > a$ as $r \to \infty$. We have also shifted the wave function in the $r > a$ region due to the delta potential not being at the origin. Defining what k is,

$$k = \sqrt{\frac{2mE_b}{\hbar^2}} \tag{34}$$

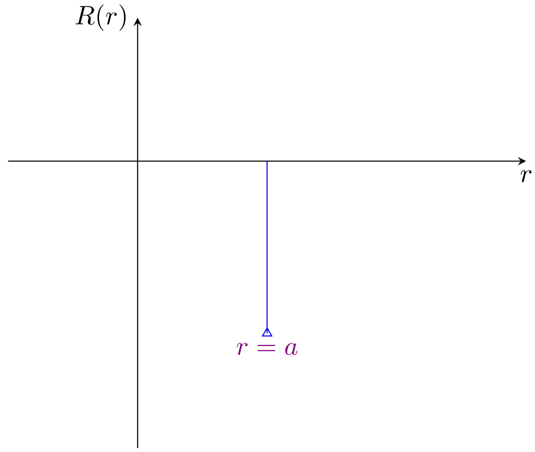

Figure 2: Attractive delta potential at $r = a$

where k has to be real to decay and not have oscillatory behavior.[4] We now apply our boundary conditions. Note, we have set $l = 0$ in the radial wave function because we are searching for β_{min} which means we look at the lowest orbital.

$$u_I(r = a) = u_{II}(r = a) \tag{35}$$

$$A \sinh(ka) = 1 \tag{36}$$

$$\Rightarrow A = \frac{1}{\sinh(ka)} \tag{37}$$

To now look at the smoothly varying condition

$$\lim_{\epsilon \to 0} \left(\partial_r u(r) \Big|_{r=a+\epsilon} - \partial_r u(r) \Big|_{r=a-\epsilon} \right) = \lim_{\epsilon \to 0} \left(-\frac{2m}{\hbar^2} \int_{a-\epsilon}^{a+\epsilon} (E - V(r)) u(r) dr \right) \tag{38}$$

$$-k - Ak \cosh(ka) = \lim_{\epsilon \to 0} \left(-\beta \int_{a-\epsilon}^{a+\epsilon} \delta(r-a) u(r) dr \right) \tag{39}$$

$$k + Ak \cosh(ka) = \beta u(a) = \beta \tag{40}$$

where we have evaluated the wave function at $r = a$ and took the limit of $\epsilon \to 0$ to arrive at 40. Plugging in for our constant A, we arrive at

$$k \left(1 + \frac{\cosh(ka)}{\sinh(ka)} \right) = \beta \tag{41}$$

$$k \left(\sinh(ka) + \cosh(ka) \right) = \beta \sinh(ka) \tag{42}$$

To find β_{min}, we need to perform $k \to 0$[5] in 42. Note, if one were asked to find the energy of the weakly bound state, one would arrive at a transcendental equation for E_{min} in using 42. Taking the limit as $k \to 0$,

$$\lim_{k \to 0} \left(k \left(\sinh(ka) + \cosh(ka) \right) = \beta \sinh(ka) \right) \tag{43}$$

$$\tag{44}$$

Using limit laws, we break up this limit and evaluate each side accordingly.

$$\lim_{k \to 0} \left(\sinh(ka) + \cosh(ka) \right) = \lim_{k \to 0} \beta \frac{\sinh(ka)}{k} \tag{45}$$

[4] This can simply derived from the S.E. in each region and remember, $E \to -|E|$ since it is a bound state.

[5] The reason being that as $k \to 0$, the potential becomes weaker and the state will become less bound until the state is not bound at all. Thus, by looking at $k \to 0$ (which is effectively looking at $E_b \to 0$), we are looking at the minimum strength our potential requires to have one bound state (for a ground state).

Looking at the LHS,

$$\lim_{k\to 0}(\sinh(ka)+\cosh(ka)) = \lim_{k\to 0}(ka+1) \approx 1 \tag{46}$$

Looking at the RHS,

$$\lim_{k\to 0}\beta\frac{\sinh(ka)}{k} = \beta\lim_{k\to 0}\frac{ka}{k} \tag{47}$$
$$= \beta a \tag{48}$$

Setting the LHS equal to the RHS

$$1 = a\beta \tag{49}$$
$$\Rightarrow \beta_{\text{min}} = \frac{1}{a} \tag{50}$$

Performing a quick check for β, such that the units of inverse-length are indeed justified, take a look at our scattering potential

$$V(r) = \frac{\hbar^2}{2m}\beta\delta(r-a) \tag{51}$$

Our potential has to have units of energy, $\hbar$ has units of $\text{eV}\cdot\text{s}$, m has units of eV/c^2, and $\delta(r)$ has units of $1/L$. Plugging all of this in,

$$V(r) = \frac{\hbar^2}{2m}\beta\delta(r-a) \longrightarrow \text{eV} = \frac{\text{eV}^2 s^2 c^2}{\text{eV}}\beta\frac{1}{L} \tag{52}$$
$$= \text{eV}s^2\frac{L^2}{s^2}\beta\frac{1}{L} \tag{53}$$
$$= \text{eV}L\beta \tag{54}$$
$$\tag{55}$$

Thus,

$$\beta = \frac{1}{L} \tag{56}$$

which signifies that 50 has the correct units.

Spring 2001, Problem 2

A particle of mass m moves under the influence of a repulsive spherically symmetric potential,

$$V(r) = \begin{cases} V_0, & r<a \\ 0, & r>a \end{cases} \longrightarrow V_0\Theta(a-r) \tag{57}$$

1. Find the s-wave phase shift $\delta(E)$ for energies $E < V_0$
2. What is the cross section for scattering in the limit $E \to 0$.

Part a

For the region $r < a$, call it region 1, the radial wave function (recalling that it is $u(r) = rR(r)$ that solves the modified SE) is

$$R(r) = \frac{u(r)}{r} \qquad u(r) = e^{\pm iqr} \tag{58}$$

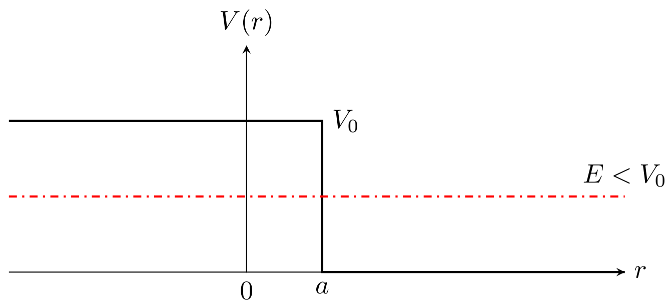

Figure 3: Step function at $r = a$

where this originated from solving the modified SE 90 for $l = 0$, which gives us 58. In the region $r < a$, the energy is less than the potential step $E < V_0$, and our wave number q becomes

$$q = \pm\sqrt{\frac{2m(E - V_0)}{\hbar^2}} \Longrightarrow q = \pm i\sqrt{\frac{2m(V_0 - E)}{\hbar^2}} \tag{59}$$

This then leads to having real exponentials for $u(r)$ which can be treated with $\sinh(qr)$ due to the wave function needing to pass through the origin.

$$R(r) = \frac{u(r)}{r} = A\frac{\sinh(qr)}{r}, \qquad q \equiv \sqrt{\frac{2m(V_0 - E)}{\hbar^2}} \tag{60}$$

This also makes sense since in the $r < a$ region the wave function must decay. For the region $r > a$, we consider the detector to be far from the range of the potential and we can use the asymptotic version of the hankel functions to define the radial solution[6].

$$R_l(r)\Big|_{r>a} = \frac{1}{2}\left(e^{2i\delta}h_l(kr) + h_l^*(kr)\right) \tag{61}$$

$$h_l(kr)\big|_{r>a} \approx \frac{(-i)^{l+1}e^{ikr}}{kr} \tag{62}$$

$$h_l^*(kr)\big|_{r>a} \approx \frac{(i)^{l+1}e^{-ikr}}{kr} \tag{63}$$

$$\Rightarrow R_{l=0}(r)\Big|_{r>a} = \frac{-ie^{2i\delta}e^{ikr} + ie^{-ikr}}{2kr} = \frac{e^{i\delta}\sin(kr + \delta)}{kr} \tag{64}$$

where

$$k = \sqrt{\frac{2mE}{\hbar^2}} \tag{65}$$

The wave function then is

$$R(r) = \begin{cases} e^{i\delta}\dfrac{\sin(kr + \delta)}{kr} & r > a \\ A\dfrac{\sinh(qr)}{r} & r < a \end{cases} \tag{66}$$

We then use the smoothly varying and continuous boundary solutions to solve for our constant and the phase.

[6]Look at additional section for alternative approach

$$R_{0,1}(r=a) = R_{0,2}(r=a) \tag{67}$$
$$A\frac{\sinh(qa)}{a} = \frac{e^{i\delta}\sin(ka+\delta)}{ka} \tag{68}$$
$$\Rightarrow A = \frac{e^{i\delta}\sin(ka+\delta)}{k\sinh(qa)} \tag{69}$$
$$\partial_r R_{0,1}(r)\bigg|_{r=a} = \partial_r R_{0,2}(r)\bigg|_{r=a} \tag{70}$$
$$\frac{qA\cosh(qa)}{a} = \frac{e^{i\delta}\cos(ka+\delta)k}{ka} \tag{71}$$
$$\frac{q\cosh(qa)}{a}\frac{e^{i\delta}\sin(ka+\delta)}{k\sinh(qa)} = \frac{e^{i\delta}\cos(ka+\delta)k}{ka} \tag{72}$$
$$\Rightarrow \delta_0 = \tan^{-1}\left(\tanh(qa)\frac{k}{q}\right) - ka \tag{73}$$

Part b

In looking at $E \to 0$, we can use the low-energy scattering cross section. Note that by looking at $E \to 0$ for our incoming beam, this simply means $k \to 0$ as defined in 65. Also, in the low-energy limit the $l = 0$ phase shift dominates[7]

$$\sigma = \frac{4\pi}{k^2}\sum_l (2l+1)\sin^2(\delta_l) \tag{74}$$
$$\Rightarrow \sigma = \frac{4\pi}{k^2}\sin^2(\delta_0) \tag{75}$$
$$\delta_0 = \tan^{-1}\left(\tanh(qa)\frac{k}{q}\right) - ka \tag{76}$$
$$\delta_0 \approx \tanh(qa)\frac{k}{q} - ka \tag{77}$$
$$\frac{4\pi}{k^2}\sin^2(\delta_0) \approx \frac{4\pi}{k^2}\sin^2\left(ka\left(\frac{\tanh(qa)}{qa} - 1\right)\right) \tag{78}$$
$$\approx \frac{4\pi}{k^2}\left(ka\left(\frac{\tanh(qa)}{qa} - 1\right)\right)^2 \tag{79}$$
$$\sigma = 4\pi a^2\left(\frac{\tanh(qa)}{qa} - 1\right)^2 \tag{80}$$

To verify that this does indeed make sense, we can look at the classical limit where $V_0 \to \infty$ ($q \to \infty$). This would represent a hard sphere with radius a.

$$q\bigg|_{V_0\to\infty} = \sqrt{\frac{2mV_0}{\hbar^2}} \tag{81}$$
$$\sigma\bigg|_{V_0\to\infty} = 4\pi a^2\left(\frac{\tanh(qa)}{qa} - 1\right)^2\bigg|_{V_0\to\infty} \approx 4\pi a^2\left(\frac{1}{qa} - 1\right)^2\bigg|_{V_0\to\infty} \tag{82}$$
$$\approx 4\pi a^2(-1)^2 = 4\pi a^2 \tag{83}$$

where we have made use of the identity

$$\tanh(x)\bigg|_{x\to\infty} \to 1 \tag{84}$$

Indeed, this does match the classical interpretation of scattering off a hard sphere.

[7] See "Additional" section.

Spring 2000, Problem 7

A particle of mass mass m moving through a normal **THREE-DIMENSIONAL** space feels a spherically symmetric attractive potential.

$$V(r) = -\beta\delta(r-R) \qquad \beta > 0 \tag{85}$$

1. For fixed R and m, find the minimum strength of the potential β that results in the existence of a bound state. Express β_{min} as a function of $\hbar$, R, and m.
2. A spherical s wave scatters off the potential with asymptotic form

$$\psi(r) \sim \frac{1}{r}\left(e^{-ikr} - e^{ikr+2i\delta}\right) \tag{86}$$

where r is the distance from the origin.

Since the question stressed that it was 3D, it is worth mentioning why. Remember, the Schödinger wave equation in 3D is

$$\left(\frac{-\hbar^2}{2m}\Delta + V(\vec{r})\right)\Psi(\vec{r},t) = i\hbar\partial_t\Psi(\vec{r},t) \rightarrow \left(\frac{-\hbar^2}{2m}\Delta + V(\vec{r})\right)\Psi(\vec{r}) = E\Psi(\vec{r}) \tag{87}$$

$$\Psi(\vec{r}) = R(r)Y_{l,m}(\theta,\phi) \tag{88}$$

where we have made the transition to the TISE because we have stationary states in this simple scenario (i.e. the potential is time independent). Since this is spherically symmetric, we only care about the radial wave function.

$$u(r) = rR(r) \tag{89}$$

$$\frac{-\hbar^2}{2m}\frac{d^2}{dr^2}u(r) + \left(V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right)u(r) = Eu(r) \tag{90}$$

where $R(r)$ is the radial solution to the wave function and $u(r)$ is the substitution we made to arrive at this form of the radial wave function. This is similar to the 1D case we have seen where it is in x since this differential equation is indeed 1D; however, the caveat being that there are factors of l that affect our wave function when $l \neq 0$. This will then affect the "suppression" we see in the partial wave expansion.

Part a

We are asked to find the minimum value of β that results in a bound state ($E < V$). In general, the usual procedure to these kinds of problems go as follows:

1. Consider whether $E > V$ or $E < V$
   1.1. In this case, it is the latter since we are looking for a bound state.
2. Define the appropriate wave functions in the region
   2.1. In $r < R$, the solution must decay and pass through the origin.
   2.2. In $r > R$, the solution must decay as $r \rightarrow \infty$.
3. Apply continuous and smoothly varying boundary conditions.
   3.1. One will have to thread carefully with the smoothly varying boundary condition due to the delta potential.
4. Since we are looking for the minimum of some quantity, expect to take the limit of something.

4.1. In this case, we are looking for β_{min}, which is related to the binding energy, or going back a layer, k.

It helps to have a mental image of the problem as well (Figure 1.) to "better define" the wave function behavior in each region.

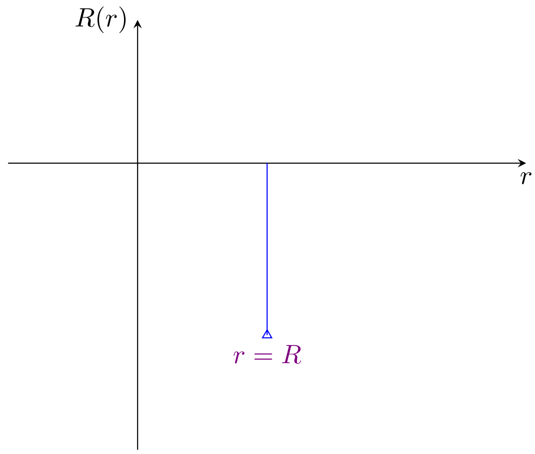

Figure 4: Attractive delta potential at $r = R$

Defining our wave functions,

$$\Psi(r) = \begin{cases} Ae^{kr} + Be^{-kr} = A\sinh kr, & r < R \\ e^{-k(r-R)}, & r > R \end{cases} \tag{91}$$

where we have imposed the condition that the wave function must pass through the origin for $r < R$ and must decay to zero in $r > R$ as $r \to \infty$. We have also shifted the wave function in the $r > R$ region due to the delta potential not being at the origin. Defining what k is,

$$k = \sqrt{\frac{2mE_b}{\hbar^2}} \tag{92}$$

where k has to be real to decay and not have oscillatory behavior.[8] We now apply our boundary conditions. Note, we have set $l = 0$ in the radial wave function because we are searching for β_{min} which means we look at the lowest orbital.

$$\Psi_I(r = R) = \Psi_{II}(r = R) \tag{93}$$

$$A\sinh kR = 1 \tag{94}$$

$$\Rightarrow A = \frac{1}{\sinh kR} \tag{95}$$

To now look at the smoothly varying condition[9]

$$\lim_{\epsilon\to 0}\left(\partial_r\Psi(r)\bigg|_{r=R+\epsilon} - \partial_r\Psi(r)\bigg|_{r=R-\epsilon}\right) = \lim_{\epsilon\to 0}\left(-\frac{2m}{\hbar^2}\int_{R-\epsilon}^{R+\epsilon}(E - V(r))\Psi(r)dr\right) \tag{96}$$

$$-k - Ak\cosh(kR) = \lim_{\epsilon\to 0}\left(-\frac{2m}{\hbar^2}\beta\int_{R-\epsilon}^{R+\epsilon}\delta(r-R)\Psi(r)dr\right) \tag{97}$$

$$k + Ak\cosh(kR) = \frac{2m}{\hbar^2}\beta\Psi(R) = \frac{2m}{\hbar^2}\beta \tag{98}$$

[8] This can simply derived from the S.E. in each region and remember, $E \to$ -$|E|$ since it is a bound state.
[9] This comes from using the reduced S.E. and integrating both sides around the delta potential.

where we have evaluated the wave function at $r = R$ and took the limit of $\epsilon \to 0$ to arrive at 98. Plugging in for our constant A, we arrive at

$$k\left(1 + \frac{\cosh(kR)}{\sinh(kR)}\right) = \frac{2m}{\hbar^2}\beta \tag{99}$$

$$k\left(\sinh(kR) + \cosh(kR)\right) = \frac{2m}{\hbar^2}\beta\sinh(kR) \tag{100}$$

To find β_{min}, we need to perform $k \to 0$[10] in 100. Note, if one were asked to find the energy of the weakly bound state, one would arrive at a transcendental equation for E_{min} in using 100. Taking the limit as $k \to 0$,

$$\lim_{k\to 0}\left(k\left(\sinh(kR) + \cosh(kR)\right) = \frac{2m}{\hbar^2}\beta\sinh(kR)\right) \tag{101}$$

$$\tag{102}$$

Using limit laws, we break up this limit and evaluate each side accordingly.

$$\lim_{k\to 0}\left(\sinh(kR) + \cosh(kR)\right) = \lim_{k\to 0}\frac{2m}{\hbar^2}\beta\frac{\sinh(kR)}{k} \tag{103}$$

Looking at the LHS,

$$\lim_{k\to 0}\left(\sinh(kR) + \cosh(kR)\right) = \lim_{k\to 0}(kR + 1) \approx 1 \tag{104}$$

Looking at the RHS,

$$\lim_{k\to 0}\frac{2m}{\hbar^2}\beta\frac{\sinh(kR)}{k} = \frac{2m}{\hbar^2}\beta\lim_{k\to 0}\frac{kR}{k} \tag{105}$$

$$= \frac{2m}{\hbar^2}\beta R \tag{106}$$

Setting the LHS equal to the RHS

$$1 = R\frac{2m}{\hbar^2}\beta \tag{107}$$

$$\Rightarrow \beta_{\text{min}} = \frac{\hbar^2}{2mR} \tag{108}$$

Part b

We are now asked to find the phase shift δ for an s wave ($l = 0$) that is induced between the incoming and outgoing wave, given the asymptotic behavior of our radial solution. We follow the outlined steps in part a and arrive at the following:

1. Consider whether $E > V$ or $E < V$
   1.1. In this case, it is the former since we are looking for the phase shift of a scattered wave.
2. Define the appropriate wave functions in the region
   2.1. In $r < R$, the solution must pass through the origin (spherical bessel functions).
   2.2. In $r > R$, the solution contains both the scattered wave and the incoming wave (asymptotic limit of hankel functions is one approach).

[10] The reason being that as $k \to 0$, the potential becomes weaker and the state will become less bound until the state is not bound at all. Thus, by looking at $k \to 0$ (which is effectively looking at $E_b \to 0$), we are looking at the minimum strength our potential requires to have one bound state (for a ground state).

3. Apply continuous and smoothly varying boundary conditions.

   3.1. Again, thread carefully with the smoothly varying boundary condition.

We already know the asymptotic limit, $r > R$, let's go ahead and look at $r < R$. Remembering that the radial solutions to $u(r)$ are

$$u(r) = rR(r) = Arj_l(kr) + Brn_l(kr) \to u(r) = Arj_0(kr) = Ar\frac{\sin(kr)}{kr} \sim A\sin(kr), \qquad r < R \tag{109}$$

We have also gotten rid of the Neuman solution as that solution blows up at the origin and we need our solution to pass through the origin. We can cast the asymptotic form of the wave function in something more familiar by realizing that $R(r) = \psi(r)$ in this case, and multiplying by r to "format" this to be $u(r)$[11]. However, for the sake of thoroughness (and study endeavors), we shall start with the general asymptotic behavior of the radial solution and work through how to arrive at the form given.

$$\psi_l(r)\Big|_{r>R} = \frac{1}{2}\left(e^{2i\delta}h_l(kr) + h_l^*(kr)\right) \tag{110}$$

$$h_l(kr)\big|_{r>R} \approx \frac{(-i)^{l+1}e^{ikr}}{kr} \tag{111}$$

$$h_l^*(kr)\big|_{r>R} \approx \frac{(i)^{l+1}e^{-ikr}}{kr} \tag{112}$$

$$\Rightarrow \psi_{l=0}(r)\Big|_{r>R} = \frac{-ie^{2i\delta}e^{ikr} + ie^{-ikr}}{2kr} = \frac{e^{i\delta}(-ie^{i(kr+\delta)} + ie^{-i(kr+\delta)})}{2kr} = \frac{e^{i\delta}\sin(kr+\delta)}{kr} \tag{113}$$

$$u(r) = r\psi(r) = \frac{e^{i\delta}\sin(kr+\delta)}{k} \sim e^{i\delta}\sin(kr+\delta), \qquad r > R \tag{114}$$

Where $h_l(kr)$ is the outgoing wave (the one with the phase) and $h_l^*(kr)$ is the incoming wave. Seeing the phase attached to the outgoing wave should make sense if one thinks about the order of events: the outgoing wave has interacted with the potential and picks up a phase; the incoming wave has not interacted with the potential so there should be no phase. Our wave functions for the incoming beam then are

$$u(r) \sim \begin{cases} A\sin(kr), & r < R \\ e^{i\delta}\sin(kr+\delta), & r > R \end{cases} \tag{115}$$

All that is left to do is apply both boundary conditions and find δ. Applying the continuous BC,

$$u(r=R)_I = u(r=R)_{II} \tag{116}$$

$$A\sin(kR) = e^{i\delta}\sin(kR+\delta) \tag{117}$$

$$\Rightarrow A = \frac{e^{i\delta}\sin(kR+\delta)}{\sin(kR)} \tag{118}$$

Applying the second BC,

$$\partial_r u(r)\Big|_{R+\epsilon} - \partial_r u(r)\Big|_{R-\epsilon} = \frac{-2m}{\hbar^2}.\lim_{\epsilon\to 0}\left(\int_{R-\epsilon}^{R+\epsilon}(E - V(r))u(r)dr\right) \tag{119}$$

When evaluating the integral over the integrand $Eu(r)$, note that as $\epsilon \to 0$, the wave functions look the same on both sides and there is no phase. Thus, integrating $Eu(r)$ over the symmetric interval would go to zero.

[11] We could've also divided our solution for $r < R$ by r as well and left the solution to $r > R$ alone.

$$\lim_{\epsilon\to 0}\left(\partial_r u(r)\Big|_{R+\epsilon} - \partial_r u(r)\Big|_{R-\epsilon}\right) = \frac{2m}{\hbar^2}\lim_{\epsilon\to 0}\left(\int_{R-\epsilon}^{R+\epsilon}(V(r))u(r)dr\right) \tag{120}$$

$$= \frac{-2m}{\hbar^2}\lim_{\epsilon\to 0}\left(\int_{R-\epsilon}^{R+\epsilon}\beta\delta(r-R)u(r)dr\right) \tag{121}$$

$$= \frac{-2m\beta}{\hbar^2}u(R) \tag{122}$$

$$e^{i\delta}\cos(kR+\delta)k - A\cos(kR)k = \frac{-2m\beta}{\hbar^2}A\sin(kR) \tag{123}$$

$$\tag{124}$$

Rearranging terms and plugging in for our constant A,

$$e^{i\delta}\cos(kR+\delta) = A\left(\cos(kR) - \frac{2m\beta}{\hbar^2 k}\sin(kR)\right) \tag{125}$$

$$e^{i\delta}\cos(kR+\delta) = \frac{e^{i\delta}\sin(kR+\delta)}{\sin(kR)k}\left(k\cos(kR) - \frac{2m\beta}{\hbar^2}\sin(kR)\right) \tag{126}$$

$$\frac{k\sin(kR)}{\left(k\cos(kR) - \frac{2m\beta}{\hbar^2}\sin(kR)\right)} = \frac{\sin(kR+\delta)}{\cos(kR+\delta)} = \tan(kR+\delta) \tag{127}$$

$$\tag{128}$$

Taking the inverse tangent of both sides and subtracting by kR, we arrive at

$$\delta_0 = \tan^{-1}\left(\frac{\sin(kR)}{\cos(kR) - \frac{2m\beta}{k\hbar^2}\sin(kR)}\right) - kR \tag{129}$$

for a s wave in a spherically symmetric potential.

Additional

Brief explanation why $l = 0$ dominates

If one really wants to know, I would recommend looking at the spherical solutions, and noting the suppression that occurs due to large l values. However, one can think of this semi-classically,

$$\vec{l} = \vec{r} \text{ x } \vec{p} \tag{130}$$

$$|\vec{l}| = rp \rightarrow |\vec{l_\infty}| = bp \tag{131}$$

where b is our impact parameter.

$$l = bp \tag{132}$$

$$b = \frac{l}{p} = \frac{l}{\hbar k} \tag{133}$$

At low energies k is small, which means our impact parameter b is large, and we will be out of range of our potential (we would have a negligible phase shift). To not "miss" our scattering center - to be in range - l has to be as small as it can be to be able to scatter from our potential. Again, if one thinks of this classically, with l representing "how far we are" from the origin of our scattering center, a smaller l means we are closer to the scattering center, meaning we will "hit it". The smallest l can go is 0, and with a $l = 0$ wave (s-wave), this would scatter the most.

Second Approach to Spring 2001, Problem 2, Part a

This approach uses the SE explicitly to solve for $u(r)$ due to the simplicity of this differential equation. We look at the modified SE in the region $r > a$ for $l = 0$

$$\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial r^2}u(r) = Eu(r) \tag{134}$$

We use a trial solution of e^{sr} and plug into 134

$$s^2 e^{sr} = -\frac{2mE}{\hbar^2}e^{sr} = -k^2 e^{sr} \tag{135}$$

$$\Rightarrow s^2 = -k^2 \tag{136}$$

$$u(r) = e^{\pm ik} \equiv A\sin(kr) + B\cos(kr) \tag{137}$$

Using the B.C. $u(r = 0) = 0$, we arrive at

$$u(r) = rR(r) = A\sin(kr) = A(e^{ikr} - e^{-ikr}) \tag{138}$$

However, we know there is a potential at $r = a$ that affects our outgoing wave function by some phase δ.

$$A\sin(kr) = A(e^{ikr} - e^{-ikr}) \rightarrow e^{i(kr+\delta)} - e^{-i(kr-\delta)} = A\sin(kr + \delta) \tag{139}$$

$$A = 1 \tag{140}$$

Our incoming wave function should not be affected by the potential since it has not yet interacted with it. This is remedied by multiplying both top and bottom by a phase phase of $e^{i\delta}$ but in pragmatic hindsight, we know the global phase factor in the denominator will disappear when looking for the modulus squared of a physical observable. Thus, we can not include it explicitly here and drop it. Our solution for both regions then are

$$u(r) = \begin{cases} e^{ikr}e^{2i\delta} - e^{-ikr} = e^{i\delta}\sin(kr + \delta) & r > a \\ A\sinh(qr) & r < a \end{cases} \tag{141}$$

Having our wave functions for each region, it is just a matter of applying both boundary conditions to find the $l = 0$ phase.

$$u_1(r = a) = u_2(r = a) \tag{142}$$
$$A \sinh(qa) = e^{i\delta} \sin(ka + \delta) \tag{143}$$
$$\Rightarrow A = \frac{e^{i\delta} \sin(ka + \delta)}{\sinh(qa)} \tag{144}$$
$$\partial_r u_1(r)\big|_{r=a} = \partial_r u_2(r)\big|_{r=a} \tag{145}$$
$$Aq \cosh(qa) = e^{i\delta} k \cos(ka + \delta) \tag{146}$$
$$\frac{e^{i\delta} \sin(ka + \delta)}{\sinh(qa)} q \cosh(qa) = e^{i\delta} k \cos(ka + \delta) \tag{147}$$
$$\Rightarrow \delta_0 = \tan^{-1}\left(\tanh(qa)\frac{k}{q}\right) - ka \tag{148}$$