FINAL EXAM

PHYSICS 851, FALL 1998

1. (15 pt.s) Consider a spin 1/2 system. The projection operator P_z projects the component of the wave function that has positive spin along the z axis.

$$\langle\eta|P_z|\eta\rangle = |\langle z,\uparrow|\eta\rangle|^2$$

   (a) Express P_z as a matrix in the basis where $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ denotes a state with positive spin along the z axis.

   (b) Write down the density matrix for a state that is an incoherent mixture of 50% positive spin along the y axis and 50% negative spin along the y axis.

   (c) If the Hamiltonian is defined as:

$$\mathcal{H} = \alpha + \beta\sigma_x$$

   Calculate the expectation of $\mathcal{H}$ for the state described in b.

$\langle \eta | P_z | \eta \rangle = |\langle \uparrow | \eta \rangle|^2$ (spin 1/2)

P_z is the $\uparrow$-projection operator

a) Express P_z as a matrix. $|\uparrow\rangle \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|\downarrow\rangle \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\boxed{P_z = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}$$

Check: let $|\eta\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$.

Then $\langle \eta | P_z | \eta \rangle = \langle \eta | \begin{pmatrix} a \\ 0 \end{pmatrix} = |a|^2$ ✓

$\langle \uparrow | \eta \rangle = (1 \;\; 0) \begin{pmatrix} a \\ b \end{pmatrix} = a \Rightarrow |\langle \uparrow | \eta \rangle|^2 = |a|^2$ ✓

(b) Write down the density matrix for a state that is an incoherent mixture of 50% positive spin along the y axis and 50% negative spin along the y axis.

b) Incoherent mixture $\Rightarrow$ 50% $\uparrow$-y & 50% $\downarrow$-y

$$\rho = \frac{1}{2}\left(|y,\uparrow\rangle\langle y,\uparrow| + |y,\downarrow\rangle\langle y,\downarrow|\right)$$

$|y,\uparrow\rangle \rightarrow \begin{pmatrix}1\\0\end{pmatrix}$, $|y,\downarrow\rangle \rightarrow \begin{pmatrix}0\\1\end{pmatrix}$ (Using y-representation)

$$\Rightarrow \boxed{\rho = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \frac{1}{2}\mathbb{1}}$$

(c) If the Hamiltonian is defined as:

$$\mathcal{H} = \alpha + \beta\sigma_x$$

Calculate the expectation of $\mathcal{H}$ for the state described in b.

c) Reminder: $\langle\Psi|H|\Psi\rangle = Tr(\rho_\Psi H)$ (Eq. 1.31 in Lecture notes)

Then $\langle H\rangle = \alpha\, Tr(\rho) + \beta\, Tr(\rho\sigma_x)$

$= \frac{\alpha}{2}(2) + \frac{\beta}{2}\, Tr(\cancel{\sigma_x}) \rightarrow 0$ reminder: $\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

$$\Rightarrow \boxed{\langle H\rangle = \alpha}$$

Fall '96 Exam Q2

2. (15 pt.s) Consider two flavors of neutrinos, the μ neutrino and the τ neutrino. Suppose that the Hamiltonian can be written as a free term plus a term that mixes the μ and τ neutrinos, which is proportional to α.

$$\mathcal{H} = \begin{pmatrix} m_\mu c^2 & 0 \\ 0 & m_\tau c^2 \end{pmatrix} + \alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

If a neutrino starts as a μ neutrino, what is the probability, as a function of time, of being a τ neutrino?

Goal: Find $P(t) = |\langle \mu | \tau(t) \rangle|^2$

First, write H in terms of $\vec{\sigma}$, $\mathbb{1}$.

Interaction part: $\alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \alpha \sigma_x$

Mass part: $\begin{pmatrix} m_\mu & 0 \\ 0 & m_\tau \end{pmatrix} c^2 = \frac{1}{2}(m_\mu c^2 + m_\tau c^2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}(m_\mu c^2 - m_\tau c^2) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$= \frac{1}{2}(m_\mu c^2 + m_\tau c^2)\, \mathbb{1} + \frac{1}{2}(m_\mu c^2 - m_\tau c^2)\, \sigma_z$$

so $H = \frac{1}{2}(m_\mu c^2 + m_\tau c^2)\, \mathbb{1} + \beta \sigma_z + \alpha \sigma_x$, $\quad \beta \equiv \dfrac{m_\mu c^2 - m_\tau c^2}{2}$

$|\mu\rangle \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\quad |\tau\rangle \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (based on masses in H)

Now apply time evolution to $|\tau\rangle$:

$$|\tau(t)\rangle = U(t)|\tau\rangle = e^{-iHt/\hbar}|\tau\rangle$$

ignore overall phase

$$= \cancel{e^{-i(m_\mu c^2 + m_\tau c^2)t/\hbar}}\; e^{-i(\beta\sigma_z + \alpha\sigma_x)t/\hbar}\, |\tau\rangle$$

$$= e^{-i\vec{v}\cdot\vec{\sigma}\, t/\hbar}\, |\tau\rangle \quad \text{where } \vec{v} = \alpha\hat{x} + \beta\hat{z}$$

Now use $e^{-i\vec{\varphi}\cdot\vec{\sigma}} = \mathbb{1}\cos\varphi - i\sin\varphi\,(\hat{\varphi}\cdot\vec{\sigma})$ (like in Eq. 1.45)

$$= \mathbb{1}\cos\omega t - i\sin\omega t\, \frac{(\alpha\sigma_x + \beta\sigma_z)}{\sqrt{\alpha^2+\beta^2}}\, |\tau\rangle \quad \text{where } \omega \equiv \frac{\sqrt{\alpha^2+\beta^2}}{\hbar}$$

$$= \left(\cos\omega t + \frac{i\beta \sin\omega t}{\sqrt{\alpha^2+\beta^2}}\right)|\tau\rangle - \frac{i\alpha}{\sqrt{\alpha^2+\beta^2}} \sin\omega t\, |\mu\rangle$$

so $\quad |\tau(t)\rangle = \left(\cos\omega t + \dfrac{i\beta \sin\omega t}{\sqrt{\alpha^2+\beta^2}}\right)|\tau\rangle - \dfrac{i\alpha}{\sqrt{\alpha^2+\beta^2}} \sin\omega t\,|\mu\rangle$

Then $\quad \langle\mu|\tau(t)\rangle = i\alpha \sin\omega t \quad \rightarrow \quad \boxed{P(t) = |\langle\mu|\tau(t)\rangle|^2 = \dfrac{\alpha^2 \sin^2\omega t}{(\alpha^2+\beta^2)}}$

Check: $\quad P_\tau(t) = |\langle\tau|\tau(t)\rangle|^2 = \cos^2\omega t + \dfrac{\beta^2}{\alpha^2+\beta^2}\sin^2\omega t$

Then $\quad P_\tau + P_\mu = \cos^2\omega t + \dfrac{\beta^2}{\alpha^2+\beta^2}\sin^2\omega t + \dfrac{\alpha^2}{\alpha^2+\beta^2}\sin^2\omega t$

$$= \cos^2\omega t + \cancelto{1}{\frac{(\alpha^2+\beta^2)}{(\alpha^2+\beta^2)}}\sin^2\omega t = \cos^2\omega t + \sin^2\omega t = 1 \checkmark$$

So there is 100% probability of either being a μ or τ.

Fall '99 Exam Q2

2. A spin 1/2 particle has positive spin along the y axis.

   (a) Write down the density matrix ρ for this state. Use the basis where $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ denotes a state with positive spin along the z axis, and $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$ refers to a particle with a positive spin projection along the y axis.

   (b) Find ρ^2.

   (c) Write down the density matrix for the state that is an incoherent mixture of 50% positive spin along the y axis and 50% negative spin along the y axis.

   (d) What is the square of this density matrix?

z-representation for spin-1/2

$$\psi_{x+} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \psi_{x-} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$
$$\psi_{y+} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \quad \psi_{y-} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix},$$
$$\psi_{z+} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \psi_{z-} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(from Wikipedia)

a) Write down density matrix ρ for $|\uparrow y\rangle$:

$$\rho = |\uparrow y\rangle\langle\uparrow y| = \frac{1}{2}(1 \;\; -i)\begin{pmatrix} 1 \\ i \end{pmatrix} = \boxed{\frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}}$$

(b) Find ρ^2.

b) $$\rho^2 = \frac{1}{4}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 & -2i \\ 2i & 2 \end{pmatrix}$$
$$= \boxed{\frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}}$$

(Shortcut: $\rho^2 = \rho$ for pure states)

(c) Write down the density matrix for the state that is an incoherent mixture of 50% positive spin along the y axis and 50% negative spin along the y axis.

c) $$\boxed{\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}\mathbb{1}}$$

(This is the same as '98 Q1B)

for incoherent mixture (50/50)

(d) What is the square of this density matrix?

d) $$\rho^2 = \frac{1}{4}\mathbb{1}\,\mathbb{1} = \frac{1}{4}\mathbb{1} = \boxed{\frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

Note: $\rho^2 \neq \rho$ here because this state is not a pure eigenstate.