Subject Exam Review: Chapter 9

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I August 2020 Problem 5

I.i The Problem

- Consider a ONE-DIMENSIONAL world, where a non-relativistic particle of mas M is in the ground state of a harmonic oscillator characterized by frequency ω_0. The harmonic oscillator is in a large box of length L which is populated by a bath of massless particles. The probability that any given state in the box is occupied is $f(k)$, where k is the wave number of the massless particle. The harmonic oscillator can be excited to the first excited state via the weak coupling,

$$V = g \int dx\ \Psi^\dagger(x) x \Psi(x) \Phi(x), \tag{1.1}$$

where Ψ is the field operator for the massive particle, with

$$[\Psi(x,t), \Psi^\dagger(x',t)] = \delta(x - x'), \tag{1.2}$$

and Φ is the field operator for the massive particle:

$$\Phi(x,t) = \sum_k \frac{1}{\sqrt{2E_k L}} [a_k e^{-i\omega t + ikx} + a_k^\dagger e^{i\omega t - ikx}], \tag{1.3}$$

where

$$[a_k, a_{k'}^\dagger] = \delta_{k,k'}. \tag{1.4}$$

Using Fermi's golden rule and the dipole approximation, find the rate at which the massive particle is excited to the first excited state from the ground state. Your answer should be in terms of $m, \omega_0, g, f(k)$.

I.ii The Approach

I don't know about you, but when I look at Equation 1.1, I find it to be eerily familiar, and I can't quite put my finger on it. Ah, of course. It's reminiscent of an interaction term of a Lagrangian; that means that we should be able to read off the coupling from it. And that's just what we've done; Figure 1.1 shows the Feynman diagram for the coupling. Note that this representation is not to be taken too literally, since this is a non relativistic system, and $\Psi^\dagger(x)$ does not represent an anti-particle; however, more importantly, if we permute the lines as we've done in Figure 1.2, this begins to look like the process in which Ψ absorbs a Φ

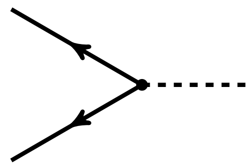

Figure 1.1: The diagrammatic representation of the coupling written in Equation 1.1; the solid lines represent the massive particle, while the dashed line corresponds to the massless particle.

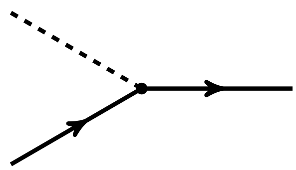

Figure 1.2: The Feynman diagram which depicts the process in which a Ψ particle absorbs a Φ particle.

particle. I hope that you have enjoyed this little aside which has gotten us no closer to the solution but which gives us a nice way of conceptualizing the process.

Now that we're past that little interlude, we can actually discuss how we're going to solve the problem: First of all, if we're to get anywhere we will, of course, need Fermi's golden rule, which allows us to write down the transition rate for the process as

$$R_{0\to 1} = \frac{2\pi}{\hbar}\sum_k f(k)|V_{10}|^2\delta(\epsilon_1 - \epsilon_0 - \epsilon_\Phi), \tag{1.5}$$

where $\epsilon_0, \epsilon_\Phi, \epsilon_1$ are the energies of the ground state harmonic oscillator, massless particle, and the first excited state harmonic oscillator, respectively (the delta function imposes energy conservation, since the expression is only non-zero when the initial and final state energies match, and $f(k)$ is the weight of the different k states), and V_{10} is shorthand for the matrix element

$$V_{10} \equiv \langle\psi_1|\, V \,|\psi_0, k_\Phi\rangle\,; \tag{1.6}$$

the states $|\psi_0, k_\Phi\rangle\,, |\psi_1\rangle$ correspond to the initial and final states, respectively. Now, we can rewrite the matrix elements by using the relations

$$\Psi(x)\,|\psi_0\rangle = \psi_0(x)\,|0\rangle\,, \tag{1.7a}$$

$$\Psi(x)\,|\psi_1\rangle = \psi_1(x)\,|0\rangle\,, \tag{1.7b}$$

where $|0\rangle$ is the vacuum state, and $\psi_0(x), \psi_1(x)$ are the ground and first-excited states, respectively. When we plug Equation 1.1 into Equation 1.6, we can drop the $a_k^\dagger$ term in Φ, since in order to have a non-zero overlap, we'll need to annihilate the k_Φ state in the ket; this also means that we will project one term out of the sum over k.

There's one little issue that I've neglected to discuss up to this point: That is that I've not addressed how the dipole approximation comes into play. In this approximation, we expand the exponentials to zeroth-order:

$$e^{ikx} \simeq 1 + \mathcal{O}(x); \tag{1.8}$$

this makes our lives significantly better, since the field operator reduces to

$$\Phi(x) \simeq \sum_k \frac{1}{\sqrt{2E_k L}}[a_k e^{-i\omega t} + a_k^\dagger e^{i\omega t}], \tag{1.8b}$$

which means that the x-integration will be much simpler. So, once we've applied Equations 1.7a,b, we will be left with

$$\begin{aligned} V_{10} &\propto \int dx \; \psi_1^*(x) x \psi_0(x) \\ &= \langle 1| \mathcal{X} |0\rangle ; \end{aligned} \tag{1.9}$$

note that $|0\rangle, |1\rangle$ are single particle states corresponding to the ground and first-excited states of the harmonic oscillator. We can rewrite the position operator in terms of single particle creation and annihilation operators:

$$\mathcal{X} = \sqrt{\frac{\hbar}{2m\omega_0}}(a + a^\dagger); \tag{1.10}$$

from this, we would have

$$\langle 1| \mathcal{X} |0\rangle = \sqrt{\frac{\hbar}{2m\omega_0}}. \tag{1.11}$$

The last step of the calculation is to plug the freshly calculated matrix element, V_{10} into Equation 1.5 to determine the transition rate for the process; then we take the continuum limit of the k_Φ sum:

$$\sum_{k_\Phi} \simeq \frac{L}{2\pi}\int dk_\Phi. \tag{1.12}$$

Computing the k_Φ integral will be fairly simple: We utilize the delta function, noting that

$$\epsilon_0 = \frac{1}{2}\hbar\omega_0, \tag{1.13a}$$

$$\epsilon_1 = \frac{3}{2}\hbar\omega_0, \tag{1.13b}$$

$$\epsilon_\Phi = \hbar c k_\Phi; \tag{1.13c}$$

then we can rewrite the delta function as

$$\delta(\epsilon_1 - \epsilon_0 - \epsilon_\Phi) = \frac{1}{\hbar c}\delta(k_\Phi - \omega_0/c), \tag{1.14}$$

and we're done.

I.iii The Execution

For starters, from everything we've written above, we can easily write down the matrix element:

$$V_{10} \simeq \frac{g}{\sqrt{2E_{k_\Phi} L}} \sqrt{\frac{\hbar}{2m\omega_0}} e^{-i\omega t}, \tag{1.15}$$

where we applied the dipole approximation, and we've already computed the x-integration; when we plug this into Equation 1.5 and rewrite the delta function, we have

$$\begin{aligned} R_{0\to 1} &= \frac{2\pi}{\hbar} \sum_{k_\Phi} f(k_\Phi) \frac{g^2}{2E_{k_\Phi} L} \left(\frac{\hbar}{2m\omega_0} \right) \frac{1}{\hbar c} \delta(k_\Phi - \omega_0/c) \\ &\simeq \frac{g^2}{2m\hbar\omega_0 c} \int_0^\infty dk_\Phi \; \frac{f(k_\Phi)}{E_{k_\Phi}} \delta(k_\Phi - \omega_0/c) \\ &= \frac{g^2}{2m\hbar\omega_0 c} \int_0^\infty dk_\Phi \; \frac{f(k_\Phi)}{\hbar c k_\Phi} \delta(k_\Phi - \omega_0/c) \\ &= \frac{g^2}{2m\hbar^2\omega_0^2 c} f(\omega_0/c); \end{aligned} \tag{1.16}$$

note that we've introduced an extra factor of two in the second line here which corresponds to the two different momentum configurations for the Φ particle, i.e. we are only concerned with the magnitude of the momentum, so it could point in either direction.