Chapter 5: Symmetries and Conservation Laws

1 Overview

1.1 Parity Operator Π

The parity operator Π flips the sign of one or more spatial coordinates. For a three-dimensional transformation this corresponds to a planar reflection of each coordinate and can be expressed as $x \to -x$, $y \to -y$, $z \to -z$. The symmetry of an operator is even under parity if its sign is conserved and odd if its sign changes.

This is especially useful in determining whether a matrix element $\langle \Psi_{l,s} | A | \Psi_{l',s'} \rangle$ vanishes (where Ψ_i are spherical wave functions). The integral of the matrix element will give zero if the overall parity is odd.

As a quick demonstration, we can consider this for some simple spherical harmonic wave functions and an even parity operator. $\langle l = 0, m = 0 | z^2 | l = 1, m = 0 \rangle$. Now $|l = 1, m = 0\rangle = \frac{1}{\sqrt{4\pi}}$ and $|l = 0, m = 0\rangle = \sqrt{\frac{3}{4\pi}} \cos\theta$ and in spherical coordinates $z = r\cos\theta$. So evaluating this matrix element results in

$$\iiint \frac{1}{\sqrt{4\pi}} r^2 \cos^2\theta \sqrt{\frac{3}{4\pi}} \cos\theta dr d\cos\theta d\phi \tag{1}$$

Looking just at the polar dependence, we have: $\int_1^{-1} \cos^3 d\cos\theta \to (-1)^4 - (1)^4 = 0$. So the matrix element disappears.

This is the case if either the operator is odd and the wave functions have the same parity or if the operator is even and the wave functions have opposite parity. The parity of the wavefunction only depends on the orbital quantum number l via $\Pi |\Psi_{l,s}\rangle = (-1)^l |\Psi_{l,s}\rangle$.

1.2 Time Reversal Operator Θ

The time reversal operator Θ preserves the coordinates and instead reverses the time variable $t \to -t$. In quantum mechanics the time reversal operator also involves taking the complex conjugate $i \to -i$. In general, operators are even or odd under time reversal unless they are a linear combination of an odd and an even operator.

2 Practice Problems (cf. Lecture Notes p. 99)

We will now cover a number of operator combinations common in Hamiltonians and determine their action under parity and time reversal.

2.1 Problem 1: $\frac{p^2}{2m}$

The quantum momentum operator is defined as $p = -i\hbar\boldsymbol{\nabla}_r$. Now, the gradient is dependent on spatial coordinates so it is odd under parity and the momentum is odd under parity. The gradient does not depend on time but the quantum time reversal operator also requires that we take the complex conjugate of the operator. So the momentum is also odd under time reversal, as we expect analogously to the classical case.

Now, in the case of this operator $\frac{p^2}{2m}$ we can think of this as $\frac{p \cdot p}{2m}$. It is easy to see this will be even under any operation, as both momentum operators will be equally affected and their dot product will be even. So $\frac{p^2}{2m}$ will be even under parity and time reversal. This applies to the square of any operator.

2.2 Problem 2: $p \cdot r$

We've already considered the momentum operator. The position operator r is odd under parity by definition and has no time dependence or complex constituent so it is even under time reversal. As a result we can consider the individual behaviour under the transformations for the overall result.

$$\Pi(p \cdot r)\Pi^{-1} \to \Pi p \Pi^{-1} \cdot \Pi r \Pi^{-1} \to (-p) \cdot (-r) \to p \cdot r \tag{2}$$

$\Rightarrow$ Even under parity

$$\Theta(p \cdot r)\Theta^{-1} \to \Theta p \Theta^{-1} \cdot \Theta r \Theta^{-1} \to (-p) \cdot r \to -(p \cdot r) \tag{3}$$

$\Rightarrow$ Odd under time reversal

2.3 Problem 3: $L \cdot p$

The angular momentum operator $L \equiv r \times p$ can be treated like the last example $p \cdot r$ since the cross product is compatible with scalar multiplication.

$$\Pi(r \times p)\Pi^{-1} \to (-r) \times (-p) \to p \cdot r \tag{4}$$

$\Rightarrow$ Even under parity, and as a side note, this is a defining property shared by all pseudovectors.

$$\Theta(r \times p)\Theta^{-1} \to r \times (-p) \to -(r \times p) \tag{5}$$

$\Rightarrow$ Odd under time reversal

So then, considering our operator at hand:

$$\Pi(L \cdot p)\Pi^{-1} \to L \cdot (-p) \to -(L \cdot p) \tag{6}$$

$\Rightarrow$ Odd under parity

$$\Theta(L \cdot p)\Theta^{-1} \to (-L) \cdot (-p) \to L \cdot p \tag{7}$$

$\Rightarrow$ Even under time reversal

3 Derivation of Time Reversal and Parity of the electromagnetic Fields and the Vector Potential

3.1 Uses of Vector Potential

$\vec{E} = -\nabla_{\vec{r}}\phi - \partial_t \vec{A}$ $\quad$ $\vec{B} = \nabla_{\vec{r}} X \vec{A}$

3.2 Magnetic field

$\vec{B} \propto \vec{I} X \vec{r} \propto \vec{v} X \vec{r}$
$\vec{B} \propto (V_y r_z - V_z r_y)\hat{x} - (V_x r_z - V_z r_x)\hat{y} + (V_x r_y - V_y r_x)\hat{z}$

3.2.1 Time Reversal

$V_i = \frac{dr_i}{dt}$
Therefore V_i will change under time reversal and r_i will not change under time reversal
thus $\Theta V_i r_i \Theta^{-1} = \Theta V_i \Theta^{-1}\ \Theta r_i \Theta^{-1} = -V_i r_i$
$\Rightarrow$ Therefore, $\Theta B \Theta^{-1}$ = -B.

3.2.2 Parity

$V_i = \frac{dr_i}{dt}$
V_i will change under parity
r_i will change under parity
thus $\Pi V_i r_i \Pi^{-1} = (-V_i)(-r_i) = V_i r_i$.
$\Rightarrow$ Therefore, $\Pi \vec{B} \Pi^{-1} = \vec{B}$.

3.3 Vector Potential $\vec{A}$

3.3.1 Time Reversal

We know that $\Theta \vec{B} \Theta^{-1}$ = -$\vec{B}$
$\Theta \nabla_{\vec{r}} X \vec{A} \Theta^{-1}$ must equal -($\nabla_{\vec{r}} X \vec{A}$)
$\Theta \nabla_{\vec{r}} X \vec{A} \Theta^{-1} = \Theta \nabla_{\vec{r}} \Theta^{-1}$ X $\Theta \vec{A} \Theta^{-1}$
$\Theta \nabla_{\vec{r}} \Theta^{-1} = \nabla_{\vec{r}}$ because $\nabla_{\vec{r}}$ is not a function of time
Therefore, for $\Theta \vec{B} \Theta^{-1}$ = -$\vec{B}$, $\Theta \vec{A} \Theta^{-1}$ must equal -$\vec{A}$

$\Theta \nabla_{\hat{r}} \Theta^{-1}$ X $\Theta A \Theta^{-1}$ = $\nabla_{\hat{r}}$ X -A = -B
$\Rightarrow$ Therefore A is odd under Time reversal.

3.3.2 Parity

We know $\Pi \vec{B} \Pi^{-1} = \vec{B}$
$\Pi \nabla_{\vec{r}} X \vec{A} \Pi^{-1}$ must equal $\nabla_{\vec{r}} X \vec{A}$
$\Pi \nabla_{\vec{r}} X \vec{A} \Pi^{-1} = \Pi \nabla_{\vec{r}} \Pi^{-1}$ X $\Pi \vec{A} \Pi^{-1}$
$\Pi \nabla_{\vec{r}} \Pi^{-1}$ = -$\nabla_{\vec{r}}$ because $\nabla_{\vec{r}}$ in a function of position
Therefore, for $\Pi \vec{B} \Pi^{-1} = \vec{B}$, $\Pi \vec{A} \Pi^{-1}$ must equal -$\vec{A}$
$\Pi \nabla_r \Pi^{-1}$ X $\Pi \vec{A} \Pi^{-1}$ = -$\nabla_{\vec{r}}$ X -$\vec{A}$ = $\vec{B}$
$\Rightarrow$ Therefore $\vec{A}$ is odd under parity

3.4 Electric Field $\vec{E}$

3.4.1 Time Reversal

$\Theta \vec{E} \Theta^{-1} = \Theta(-\nabla_{\vec{r}}\phi - \partial_t \vec{A})\Theta^{-1}$
$\Theta(-\nabla_{\vec{r}}\phi)\Theta^{-1}$ - $\Theta \partial_t \vec{A} \Theta^{-1}$
$\Theta(-\nabla_{\vec{r}})\Theta^{-1}\ \Theta\phi\Theta^{-1}$- $\Theta\partial_t\Theta^{-1}\Theta\vec{A}\Theta^{-1}$
$\Theta(-\nabla_{\vec{r}})\Theta^{-1} = (-\nabla_{\vec{r}})$ because $(-\nabla_{\vec{r}})$ is not a function of time
$\Theta\phi\Theta^{-1} = \phi$ because, "In fact, for all Lorentz four-vectors, the three spatial components must have the opposite behavior under time reversal as the "zeroth" component" (Scott Pratt)
$\Theta\partial_t\Theta^{-1}$ = -∂_t because ∂_t is a function of time
$\Theta\vec{A}\Theta^{-1}$ = -$\vec{A}$
$\Theta \vec{E} \Theta^{-1} = \Theta(-\nabla_{\vec{r}}\phi - \partial_t \vec{A})\Theta^{-1} = (-\nabla_{\vec{r}}\phi - \partial_t \vec{A}) = \vec{E}$
$\Rightarrow$ Therefore E even under Time reversal

3.4.2 Parity

$\Pi \vec{E} \Pi^{-1} = \Pi(-\nabla_{\vec{r}}\phi - \partial_t \vec{A})\Pi^{-1}$
$\Pi(-\nabla_{\vec{r}}\phi)\Pi^{-1}$ - $\Pi\partial_t\vec{A}\Pi^{-1}$
$\Pi(-\nabla_{\vec{r}})\Pi^{-1}\ \Pi\phi\Pi^{-1}$- $\Pi\partial_t\Pi^{-1}\ \Pi\vec{A}\Pi^{-1}$
$\Pi(-\nabla_{\vec{r}})\Pi^{-1} = (\nabla_{\vec{r}})$ because $(\nabla_{\vec{r}})$ is a function of space
$\Pi\phi\Pi^{-1} = \phi$ because each part of the equation need to be changed by the same sign and the second half did change sign
$\Pi\partial_t\Pi^{-1} = \partial_t$ because ∂_t is not a function of space
$\Pi\vec{A}\Pi^{-1}$ = -$\vec{A}$
$\Pi \vec{E} \Pi^{-1} = \Pi(-\nabla_{\vec{r}}\phi - \partial_t \vec{A})\Pi^{-1} = (\nabla_{\vec{r}}\phi + \partial_t \vec{A})$ = -$\vec{E}$
$\Rightarrow$ Therefore E odd under parity

4 Summary: Parity and Time Reversal of Important Operators

The behaviour of a specific operator under parity and time reversal depends on their relation with the transformed coordinate and time respectively. The following table shows how frequently occurring operators transform.

Operator		**Parity Π**	**Time Reversal Θ**
Time	t	t (even)	-t (odd)
Position Vector	$\vec{r}$	$-\vec{r}$ (odd)	$\vec{r}$ (even)
Velocity Vector	$\vec{v} = \partial_t \vec{r}$	$-\vec{v}$ (odd)	$-\vec{v}$ (odd)
Momentum	$\vec{p} = -i\hbar \boldsymbol{\nabla}_{\vec{r}}$	$-\vec{p}$ (odd)	$-\vec{p}$ (odd)
Angular Momentum	$\vec{L} = \vec{r} \times \vec{p}$	$\vec{L}$ (even)	$-\vec{L}$ (odd)
Electric Potential	Φ	Φ (even)	Φ (even)
Vector Potential	$\vec{A}$	$-\vec{A}$ (odd)	$-\vec{A}$ (odd)
Electric field	$\vec{E} = -\boldsymbol{\nabla}_{\vec{r}}\Phi - \partial_t \vec{A}$	$-\vec{E}$ (odd)	$\vec{E}$ (even)
Magnetic Field	$\vec{B} = \boldsymbol{\nabla}_{\vec{r}} \times \vec{A}$	$\vec{B}$ (even)	$-\vec{B}$ (odd)
Electric Current Density	$\vec{j}$	$-\vec{j}$ (odd)	$-\vec{j}$ (odd)
Charge Density	ρ	ρ (even)	ρ (even)