Chapter 4

Angular Momentum

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Angular momentum operator

- Total angular momentum $\vec{J} \equiv \vec{L} + \vec{S}$
  - Orbital angular momentum $\vec{L}$
  - Spin angular momentum $\vec{S}$
- Orbital angular momentum $\vec{L}$
  - $L_z = -i\hbar(x\partial_y - y\partial_x) = -i\hbar\partial_\phi$
  - $\left|\vec{L}\right|^2 = L_x^2 + L_y^2 + L_z^2$ is a scalar
- Raising and lowering operators for angular momentum $L_\pm$
  - $L_\pm \equiv L_x \pm iL_y$

Angular momentum operator continued

- Commutation
  - L_z commutes with H if H is invariant to rotations about the z axis
  - $[L_z, L_\pm] = \pm\hbar L_\pm$
- Eigenstates
  - Can define eigenstates of a spherically symmetric H that are also eigenstates of both L^2 and L_z.
  - Eigenvalues defined in terms of m and l
    - $L_z |l, m\rangle = m\hbar |l, m\rangle$
    - $L^2 |l, m\rangle = l(l+1)\hbar^2 |l, m\rangle$

Problem

1. Consider three spin operators $\mathbf{S_x}$, $\mathbf{S_y}$ and $\mathbf{S_z}$. Circle the operators that commute with $\mathbf{S_z}$.
   1. $\mathbf{S_x}$
   2. $\mathbf{S_z}$
   3. $\mathbf{S_x^2}$
   4. $\mathbf{S_z^2}$
   5. $\mathbf{S_x^2} + \mathbf{S_y^2} + \mathbf{S_z^2}$
2. Consider two sets of spin operators, $\mathbf{S_x}$, $\mathbf{S_y}$, $\mathbf{S_z}$ and $\mathbf{L_x}$, $\mathbf{L_y}$, $\mathbf{L_z}$. You can assume $\vec{\mathbf{S}}$ operates on intrinsic spin and that $\vec{\mathbf{L}}$ describes orbital angular momentum. Circle the operators that commute with $\mathbf{S_z}$.
   1. $\mathbf{L_x}$
   2. $\mathbf{L_z}$
   3. $\mathbf{L_x^2}$
   4. $\mathbf{L_z^2}$
   5. $\mathbf{L_x^2} + \mathbf{L_y^2} + \mathbf{L_z^2}$
3. Now consider the operators $\vec{\mathbf{J}} \equiv \vec{\mathbf{L}} + \vec{\mathbf{S}}$. Circle the operators that commute with $\mathbf{S_z}$.
   1. $\mathbf{J_x}$
   2. $\mathbf{J_z}$
   3. $\mathbf{J_x^2}$
   4. $\mathbf{J_z^2}$
   5. $\mathbf{J_x^2} + \mathbf{J_y^2} + \mathbf{J_z^2}$

Solution

1. Circle the operators that commute with $\mathbf{S_z}$.
   1. $\mathbf{S_x}$
   2. $\mathbf{S_z}$
   3. $\mathbf{S_x^2}$
   4. $\mathbf{S_z^2}$
   5. $\mathbf{S_x^2 + S_y^2 + S_z^2}$
2. Circle the operators that commute with $\mathbf{S_z}$.
   1. $\mathbf{L_x}$
   2. $\mathbf{L_z}$
   3. $\mathbf{L_x^2}$
   4. $\mathbf{L_z^2}$
   5. $\mathbf{L_x^2 + L_y^2 + L_z^2}$
3. Circle the operators that commute with $\mathbf{S_z}$.
   1. $\mathbf{J_x}$
   2. $\mathbf{J_z}$
   3. $\mathbf{J_x^2}$
   4. $\mathbf{J_z^2}$
   5. $\mathbf{J_x^2 + J_y^2 + J_z^2}$

Details

Notation

- Define “vector" of operators: $\vec{\mathbf{S}} = \mathbf{S_x}\hat{x} + \mathbf{S_y}\hat{y} + \mathbf{S_z}\hat{z}$ with $\mathbf{S^2} = \mathbf{S_x^2} + \mathbf{S_y^2} + \mathbf{S_z^2}$
- Total angular momentum: $\vec{\mathbf{J}} \equiv \vec{\mathbf{L}} + \vec{\mathbf{S}}$; $\mathbf{J_i} = \mathbf{L_i} + \mathbf{S_i}$; $\mathbf{J^2} = \mathbf{L^2} + \mathbf{S^2} + 2\mathbf{L} \cdot \mathbf{S} \neq \mathbf{L^2} + \mathbf{S^2}$

Commutator properties:

- $[\mathbf{A}, \mathbf{B}] = -[\mathbf{B}, \mathbf{A}]$
- $[\mathbf{A} + \mathbf{B}, \mathbf{C}] = [\mathbf{A}, \mathbf{C}] + [\mathbf{B}, \mathbf{C}]$
- $[\mathbf{AB}, \mathbf{C}] = \mathbf{A}[\mathbf{B}, \mathbf{C}] + [\mathbf{A}, \mathbf{C}]\mathbf{B}$

Important commutation relations (hold for **L** as well):

- $[\mathbf{S_i}, \mathbf{S_j}] = i\epsilon_{ijk}\mathbf{S_k}$
- $[\mathbf{S^2}, \mathbf{S_i}] = 0$
- $[\mathbf{S_i}, \mathbf{L_j}] = 0$; $[\mathbf{S_i^{(1)}}, \mathbf{S_j^{(2)}}] = 0$; $[\mathbf{S_i}, \mathbf{J_j}] = [\mathbf{S_i}, \mathbf{S_j}]$

Addition of Angular Momentum

Suppose we have two particles with angular momenta $\hat{J}_1$ and $\hat{J}_2$. Then, we have:

$$\left[\hat{J}_{nx}, \hat{J}_{ny}\right] = i\hbar \hat{J}_{nz}, \quad \text{etc.}, \quad \left[\hat{J}_n{}^2, \hat{J}_{ni}\right] = 0 \quad ,$$

and

$$\left[\hat{J}_{1i}, \hat{J}_{2k}\right] = 0, \quad i, k = x, y, z$$

Therefore, the four operators $\hat{J}_1^2, \hat{J}_{1z}, \hat{J}_2^2, \hat{J}_{2z}$ constitute a set of compatible observables. (j_1, j_2, m_1, m_2) can be mutually observed... They should have a common set of eigenbasis:

$$\hat{J}_1^2 \left|j_1, m_1, j_2, m_2\right\rangle = j_1 \left(j_1 + 1\right) \hbar^2 \left|j_1, m_1, j_2, m_2\right\rangle$$

$$\hat{J}_{1z} \left|j_1, m_1, j_2, m_2\right\rangle = m_1 \hbar \left|j_1, m_1, j_2, m_2\right\rangle$$

$$\hat{J}_2^2 \left|j_1, m_1, j_2, m_2\right\rangle = j_2 \left(j_2 + 1\right) \hbar^2 \left|j_1, m_1, j_2, m_2\right\rangle$$

$$\hat{J}_{2z} \left|j_1, m_1, j_2, m_2\right\rangle = m_2 \hbar \left|j_1, m_1, j_2, m_2\right\rangle$$

Addition Theorem

Now, consider the operator $\hat{J} = \hat{J}_1 + \hat{J}_2$.
The allowed eigenvalues of this operator $\hat{J}$ are:

$$j = j_1 + j_2, j_1 + j_2 - 1, \ldots, |j_1 - j_2| .$$

And the eigenvalues of the corresponding $\hat{J}_z$ are: $m = j, j-1, \ldots, -j$
We can see that $\left[\hat{J}^2, \hat{J}^2{}_n\right] = 0$, $\left[\hat{J}^2, \hat{J}_{zn}\right] \neq 0$, and $\left[\hat{J}_z, \hat{J}^2{}_n\right] = 0$.
Thus, the four operators $\hat{J}^2, \hat{J}_z, \hat{J}_1^2, \hat{J}_2^2$ constitute a set of compatible observables.
(j, m, j_1, j_2) can be mutually observed... They should have a common set of eigenbasis:

$$\hat{J}^2 |j, m, j_1, j_2\rangle = j(j+1)\hbar^2 |j, m, j_1, j_2\rangle$$
$$\hat{J}_z |j, m, j_1, j_2\rangle = m\hbar |j, m, j_1, j_2\rangle$$
$$\hat{J}_1^2 |j, m, j_1, j_2\rangle = j_1 (j_1 + 1) \hbar^2 |j, m, j_1, j_2\rangle$$
$$\hat{J}_2^2 |j, m, j_1, j_2\rangle = j_2 (j_2 + 1) \hbar^2 |j, m, j_1, j_2\rangle$$

Clebsch-Gordon Coefficients

- For a system of two spin-1/2 particles, we start in the highest j, m state of the coupled basis which describes the total angular momentum of both particles $J = J_1 + J_2$ where $J_1 = L_1 + S_1$, the orbital and spin components of angular momentum.
- There is only one possible uncoupled basis state for our initial state, so $\left|j, m_j\right\rangle = |s_1 = 1/2, s_2 = 1/2, m_1 = 1/2, m_2 = 1/2\rangle = |\uparrow\uparrow\rangle$
- To generate other coupled basis states, we act on this state with the lowering operator:
  $J_- \ \left|j, m_j\right\rangle = \hbar\sqrt{j\,(j+1) - m_j\,(m_j - 1)}\,\left|j, m_j - 1\right\rangle$
- The corresponding uncoupled basis states are found by similarly acting on the initial uncoupled state by:

$$(S_{1-} + S_{2-})\ |m_1, m_2\rangle = S_{1-}\ |m_1, m_2\rangle + S_{2-}\ |m_1, m_2\rangle$$

From completeness:

$$\mathbb{I} = \sum_{m_1, m_2} |s_1, s_2, m_1, m_2\rangle \langle s_1, s_2, m_1, m_2|$$

$$|j, m_j\rangle = \sum_{m_1, m_2} |s_1, s_2, m_1, m_2\rangle \langle s_1, s_2, m_1, m_2 \,|\, j, m_j\rangle$$

$$= \sum_{m_1, m_2} \underbrace{\langle j, m_j \,|\, s_1, s_2, m_1, m_2\rangle}_{\text{Clebsch-Gordon coefficients}} |s_1, s_2, m_1, m_2\rangle$$

In this case, the first lowering operation always gives:

$$J_- \,|j, m_j\rangle = (S_{1-} + S_{2-})\,|m_1, m_2\rangle$$

$$\hbar\sqrt{j(j+1) - m_j(m_j - 1)}\,|j, m_j - 1\rangle = \hbar\sqrt{m_1(m_1+1) - m_1(m_1-1)}|m_1 - 1, m_2\rangle + \hbar\sqrt{m_2(m_2+1) - m_2(m_2-1)}\,|m_1, m_2 - 1\rangle$$

$$|j, m_j - 1\rangle = \frac{\sqrt{m_1(m_1+1) - m_1(m_1-1)}}{\sqrt{j(j+1) - m_j(m_j-1)}}\,|m_1 - 1, m_2\rangle + \frac{\sqrt{m_2(m_2+1) - m_2(m_2-1)}}{\sqrt{j(j+1) - m_j(m_j-1)}}\,|m_1, m_2 - 1\rangle$$

Decay of a neutral ρ^0 with initial angular momentum

One decay mode for a neutral rho meson ρ^0 is to an electron and positron pair with branching ratio of $\sim 5 \times 10^{-5}$:

$$e^+ \leftarrow \rho^0 \rightarrow e^-$$

Suppose the rho meson was known to be in the $l = 0$ state before the decay. What is the resulting state of the electron-positron system after the decay?

- Total angular momentum is conserved, we relate the initial coupled basis state $|j = 1, m_j = 0\rangle$ to our final state in the uncoupled $|s_1, s_2, m_1, m_2\rangle$ basis. The corresponding uncoupled basis state is given by the Clebsch-Gordon coefficients:
- Note that due to spin conservation, states with $l \neq 0$ are not possible. Therefore, the angular momentum of the resulting decay is composed entirely from spin angular momentum.

Initial state is:

$$|l = 0, s = 1, m_l = 0, m_s = 0\rangle = |j = 1, m_j = 0\rangle$$

We relate this to the uncoupled e^+, e^- basis by:

$$|j = 1, m_j = 0\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$