Chapter 3: Charged Particles in Electromagnetic Fields

(Dated: December 2021)

I. INTRODUCTION

Chapter 3 considers the effects of added electric and magnetic fields on the motion of a free particle. Certain example systems closely parallel the resulting dynamics found from a classical approach. Specifically, the Heisenberg representation allows us to derive and solve the equation of motion for the position operator, $\vec{r}$, identifying the conjugate momentum. We briefly review a number of important concepts for the quantum mechanical description, as well as from classical mechanics and electromagnetism. Two notable effects, the existence of Landau levels and generation of a drift velocity, are covered as a previous exam problem.

II. RELEVANT CONCEPTS FROM THE CHAPTER

A. Electric and Magnetic Fields, Potential Four Vector

We consider the effects of added electric and magnetic fields on the motion of a free particle of charge e, mass m. The electric and magnetic field are described in terms of the scalar and vector potential:

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}, \tag{1}$$

$$\vec{B} = \nabla \times \vec{A}. \tag{2}$$

Here, the vector potential, $\vec{A}$, and scalar potential, Φ form the relativistic potential four-vector: $(\Phi, A_x, A_y, A_z)^T$. To refresh, in a new frame moving with velocity v_o along the y-dimension, the potential four vector is observed as:

$$\begin{bmatrix}\Phi' \\ A'_x \\ A'_y \\ A'_z\end{bmatrix} = \begin{bmatrix}\gamma & 0 & -\beta\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\beta\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\Phi \\ A_x \\ A_y \\ A_z\end{bmatrix}. \tag{3}$$

Here, $\beta = v_o/c$, and $\gamma = 1/\sqrt{1-\beta^2}$. Recall that physical laws are invariant when viewed from different inertial frames.

There is an apparent freedom in the choice of $\vec{E}$ and $\vec{B}$ as presented above. Rewriting as,

$$\vec{E} = -\nabla\left(\Phi - \frac{1}{c}\frac{\partial\Lambda(\vec{r},t)}{\partial t}\right) - \frac{1}{c}\frac{\partial}{\partial t}(\vec{A} + \nabla\Lambda(\vec{r},t)), \tag{4}$$

$$\vec{B} = \nabla \times (\vec{A} + \nabla\Lambda(\vec{r},t)), \tag{5}$$

the fields remain unchanged by the additional components. This is known as a gauge transformation, allowing us to rewrite $\vec{A}$ and Φ in an equivalent form, usually with some symmetry exploited in problem solving.

B. Mini-Example: Fall 2020 Midterm 2, Problem 1, Part (a)

"A particle of mass m and charge e feels a strong constant electric field $\vec{E} = E_o\hat{x}$. Additionally, the particle experiences a constant weak magnetic field $\vec{B} = B_o\hat{z}$. The magnetic field strength is less than the electric field strength, $E_o > B_o$."

(a) Describe a frame where either $\vec{E}$ or $\vec{B}$ are zero, and describe the subsequent, observed motion.

As $E_o > B_o$, we can find a frame where the magnetic field strength is zero. Assuming $\vec{A}$ lies along the y axis, we see $E_o = -\partial(-E_o x)/\partial x$, and $B_o = \partial(B_o x)/\partial x$. Our vector potential is then, $(-E_o x, 0, B_o x, 0)^T$. Considering a frame moving along the y dimension at v_o:

$$\Phi' = \gamma\Big(- E_o x - \frac{v_o}{c} B_o x\Big) \tag{6}$$

$$A'_x = A_x \tag{7}$$

$$A'_y = \gamma\Big(\frac{v_o}{c} E_o x + B_o x\Big) \tag{8}$$

$$A'_z = A_z \tag{9}$$

Setting $\vec{B}' = \nabla \times \vec{A}' = 0$ yields the speed $v_o = -B_o c/E_o$. Note that this solution would be nonphysical if $B_o > E_o$. Solving for $\vec{E}'$:

$$\vec{E}' = -\nabla\Phi = -\gamma\nabla\Big(- E_o x + \frac{B_o^2}{E_o} x\Big) = \gamma\Big(E_o - \frac{B_o^2}{E_o}\Big)\hat{x} \tag{10}$$

In the frame moving with v_o, a particle appears to accelerate in the x direction. This motion is due to $\vec{E}'$, as $\vec{B}' = 0'$.

C. Introducing Electromagnetism in Quantum Mechanics

We are interested in incorporating the potential four vector into the Hamiltonian. A complete derivation will require quantum electrodynamics. As in classical mechanics, we will just assume the form:

$$H = \frac{1}{2m}\left(\vec{P} - \frac{e\vec{A}}{c}\right)^2 + e\Phi(\vec{r}, t). \tag{11}$$

The above equation can also be obtained through a 'minimal coupling' as described in the lecture notes, noting the symmetry between the spatial partial derivatives with $\vec{A}$, and the partial time derivative with Φ.

The results from this chapter rely on the equations of motion describing a particle's trajectory. The Heisenberg representation is then a natural choice to describe our time varying operators. For an operator $A^{(H)}$ in the Heisenberg representation, we have:

$$\frac{dA^{(H)}}{dt} = \frac{\partial U^\dagger}{\partial t} A^{(S)} U + U^\dagger A^{(S)} \frac{\partial U}{\partial t} \tag{12}$$

$$= \frac{i}{\hbar}\Big(U^\dagger H U U^\dagger A^{(S)} U - U^\dagger A^{(S)} U U^\dagger H U \Big) \tag{13}$$

$$= \frac{i}{\hbar}[U^\dagger H U, A^{(H)}] \tag{14}$$

$$= \frac{i}{\hbar}[H, A^{(H)}] = \frac{i}{\hbar} e^{iHt/\hbar}[H, A^{(S)}] e^{-iHt/\hbar}. \tag{15}$$

Here, $A^{(S)}$ is the operator in the Schrodinger representation, and U is the evolution operator. We've assumed U commutes with H, and that A has no explicit time dependence. The identity was inserted in the second line. The equation of motion for the position operator is then (dropping the superscript):

$$\frac{dr_i}{dt} = \frac{i}{\hbar} e^{iHt/\hbar}[H, r_i] e^{-iHt/\hbar} \tag{16}$$

$$= \frac{1}{2mi\hbar} \sum_j \left(r_i \left(P_j - \frac{eA_j}{c} \right) \left(P_j - \frac{eA_j}{c} \right) - \left(P_j - \frac{eA_j}{c} \right) \left(P_j - \frac{eA_j}{c} \right) r_i \right). \tag{17}$$

The evolution operators have been dropped in the second line, implying the right hand side is to be taken in the Heisenberg representation. The coordinate r_i will commute with functions of p_j or r_j such that, $j \neq i$:

$$\frac{dr_i}{dt} = \frac{1}{2mi\hbar} \left(r_i \left(P_i - \frac{eA_i}{c} \right) \left(P_i - \frac{eA_i}{c} \right) + i\hbar \left(P_i - \frac{eA_i}{c} \right) - \left(P_i - \frac{eA_i}{c} \right) r_i \left(P_i - \frac{eA_i}{c} \right) \right), \tag{18}$$

$$\frac{dr_i}{dt} = \frac{(P_i - eA_i/c)}{m} = \frac{\Pi_i}{m}. \tag{19}$$

Finally, the conjugate momentum, $\vec{\Pi}$ is derived. The quantity is directly proportional to the velocity of the particle (not the operator P_i). If not obvious, the above commutator can be derived using a test function.

III. A LONGER EXAMPLE: FALL FINAL 2019, PROBLEM 7

"A positively charged particle of mass m and charge e is placed in a region with uniform magnetic field B pointing along the z axis."

(a) "Write the vector potential that describes the field such that $\vec{A}$ is in the y direction."

Following the previous example, take the four vector potential, $(0, 0, B_o x, 0)^T$.

(b) "What is the ground state energy? What is the general form for all eigen energies?"

Begin by expressing H in terms of the chosen gauge:

$$H = \frac{P_z^2}{2m} + \frac{P_x^2}{2m} + \frac{1}{2m}\left(P_y - \frac{eBx}{c}\right)^2. \tag{20}$$

Here, P_i is the momentum operator for the ith dimension. H is a function of only the spatial coordinate x, suggesting,

$$[P_y, H] = [P_z, H] = 0. \tag{21}$$

The operators P_y and P_z represent constants of motion (remember the equation of motion for operators in the Heisenberg representation), and can be replaced by their eigenvalues, $\hbar k_y$ and $\hbar k_z$. Our spatial wave function can then be taken as:

$$\psi(x,y,z) = \phi(z)\phi(y)\phi(x) = e^{ik_z z}e^{ik_y y}\phi(x), \tag{22}$$

Where $\phi(z)$ and $\phi(y)$ are momentum eigenstates. $\phi(z)$ is a simultaneous eigenstate of H, corresponding to the energy of a free particle moving unaffected in the $\hat{z}$ direction:

$$H\phi(z) = E_z\phi(z) = \frac{P_z^2}{2m}\phi(z) \longrightarrow \frac{\hbar^2 k_z^2}{2m} = E_z. \tag{23}$$

Separately, we now focus on the coupled motion in the x-y plane. As H has no explicit time dependence ($H\psi = E\psi$):

$$E\psi(x,y,z) - E_z\psi = -\frac{\hbar^2}{2m}\partial_x^2\psi + \frac{1}{2m}\left(\hbar k_y - \frac{eBx}{c}\right)^2\psi, \tag{24}$$

$$E\phi(x) - E_z\phi(x) = -\frac{\hbar^2}{2m}\partial_x^2\phi(x) + \frac{1}{2m}\left(\hbar k_y - \frac{eBx}{c}\right)\cdot\left(\hbar k_y - \frac{eBx}{c}\right)\phi(x). \tag{25}$$

Simply pulling out the factors scaling x,

$$E\phi(x) - E_z\phi(x) = -\frac{\hbar^2}{2m}\partial_x^2\phi(x) + \frac{e^2B^2}{2mc^2}\left(\frac{\hbar k_y c}{eB} - x\right)^2\phi(x), \tag{26}$$

This form is identified as the Hamiltonian of a 1D harmonic oscillator (HO), centered on x_o. The angular frequency is $\omega = eB/mc$, the cyclotron frequency. The center position, $x_o = \hbar k_y c/eB$. Though the motion occurs in the two dimensional x-y plane, the x and y coordinates are coupled, representing motion confined to a circle. We can consider the speed in the y-dimension:

$$v_y = \frac{\Pi_y}{m} = \left(\hbar k_y - \frac{eBx}{c}\right)\frac{1}{m}. \tag{27}$$

To describe the HO motion suggested by the Hamiltonian, we choose a solution, $x = x_0 + R\cos(wt)$. R is the radius of the circular path, set by initial conditions. Substituting into the above yields, $v_y = -\omega R\cos(\omega t)$. Differentiating or integrating these respective equations gives a complete description of circular motion in the x-y plane. This constraint generates degeneracy between states, and allows us to represent the energy eigenvalues of the system in terms of the 1D HO solution:

$$E_n(k_z) = \frac{\hbar^2 k_z^2}{2m} + \left(n + \frac{1}{2}\right)\hbar\omega, \tag{28}$$

$$E_0(k_z) = \frac{\hbar^2 k_z^2}{2m} + \frac{\hbar\omega}{2} \tag{29}$$

There is no dependence of energy on k_y, reflecting the degeneracy described. An unconfined particle can take any value of k_y for the same n. This group of solutions corresponds to the nth Landau level. Also note, if $h\omega$ is large with respect to the kinetic energy contributed by translational motion in the z-direction, the quantization becomes more prominent.

(c) "An electric field **E** ($|\mathbf{E}| < |\mathbf{B}|$) is added in the x direction. If the particle is initially at $x = y = 0$ at time $t = 0$ and if the initial velocity $\vec{v}(t = 0) = 0$, find its approximate position after a long time t. By "approximate", ignore any oscillatory forms to its position vs time."

We assume the same form for ψ as part (b). The Hamiltonian now reflects the added scalar potential:

$$H = \frac{\hbar^2 k_y^2}{2m} + \frac{P_x^2}{2m} + \frac{1}{2m}\left(\hbar k_y - \frac{eBx}{c}\right)^2 - eEx. \tag{30}$$

Expanding the squared term and completing the square:

$$H = \frac{\hbar^2 k_y^2}{2m} + \frac{P_x^2}{2m} + \frac{e^2 B^2}{2mc^2}\left(x - x_o\right)^2 - \frac{mc^2 E^2}{2B^2} - \frac{\hbar k_y E}{B}, \tag{31}$$

$$x_o = \frac{\hbar c_y}{eB} + \frac{mc^2 E}{eB^2}. \tag{32}$$

A particle would undergo circular motion about this shifted central position. Additionally,

$$\frac{\Pi_y}{m} = v_y = \frac{\hbar k_y}{m} - \frac{eBx}{mc} \tag{33}$$

$$\bar{v}_y = \frac{\hbar k_y}{m} - \frac{eBx_o}{mc} \tag{34}$$

$$= \frac{\hbar k_y}{m} - \frac{\hbar k_y}{m} - \frac{Ec}{B} \tag{35}$$

Ignoring oscillatory effects, the particle's position at a time t is, $y(t) = -Ect/B$. The x component of velocity averages to zero, all components of velocity average to constants, and there is no extended, translational acceleration. The result rotates with $\vec{E}$ in the x-y plane, suggesting $\vec{E}$ rotated by some angle adjusts the orientation of the drift velocity accordingly.