PHY 851 Final Exam Preparation

Chapter 6

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December 8, 2021

1 Background

1.1 Fermi's Golden Rule

Fermi's Golden Rule is a powerful method to calculate transition rates from one energy eigenstate of a Hamiltonian to another energy eigenstate induced by a perturbation. To do so, the perturbation is considered to be "small"/"weak" applied "slowly" so that first-order time dependent perturbation theory applies. Under these conditions, it can be shown that the transition rate is independent of time and is given by

$$\Gamma_{i \to f} = \frac{2\pi}{\hbar} \left| V_{fi} \right|^2 \delta(E_f - E_i) \tag{1}$$

When this expression is used, there should be a (near)-continuum of final states, which allows one to integrate over final states using the density of states to obtain

$$\Gamma_{i \to f} = \frac{2\pi}{\hbar} \left| V_{fi} \right|^2 \rho(E_f) \tag{2}$$

In the case that time-dependence must be considered for a perturbative potential (such as a harmonic oscillator) then the energy states will differ by the energy intervals for the given perturbative time-dependent potential. Continuing with the case of the harmonic perturbative potential, we know that energy states of the harmonic perturbative potential are separated by $\hbar\omega$ and the energy states described.

2 Problem: Fall 2019 Final Question 4

In one dimension, a particle of type a and mass m is in the ground state of an attractive potential

$$V_o = -\beta\delta(x)$$

A perturbative potential V_{ab} is added,

$$V_{ab} = \alpha \cos(\omega t)$$

where α is small and $\hbar\omega$ is larger than the binding energy. This converts the particle to a type b particle, which has the same mass m but does not feel the effects of V_o.

1. What is the binding energy of the a particle?
2. What is the decay rate?

2.1 Solution

2.1.1 Part 1

Given the presence of a potential in the form of a delta function, we know that the solutions to Schrödinger's Equation will have the form of

$$\psi(x) = \begin{cases} e^{qx} & \text{if } x < 0 \\ e^{-qx} & \text{if } x > 0 \end{cases}$$

In order to solve for the binding energy B, we must taking into account the boundary conditions.

$$\begin{aligned} -\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\psi_+(0) - \frac{\partial}{\partial x}\psi_-(0)\right) &= -\beta\psi(0) \\ -\frac{\hbar^2}{2m}(-q-q) &= -(-\beta) \\ \frac{\hbar^2 q}{m} &= \beta \\ q &= \frac{m\beta}{\hbar^2} \end{aligned}$$

The binding energy B is thus

$$\begin{aligned} B &= \frac{\hbar^2}{2m}q^2 \\ &= \frac{\hbar^2}{2m}\frac{m^2\beta^2}{\hbar^4} \\ &= \frac{m\beta^2}{2\hbar^2} \end{aligned}$$

2.2 Part 2

We know that Fermi's Golden Rule is given by

$$\Gamma_{i\to f} = \sum_k \frac{2\pi}{\hbar}\left|V_{fi}\right|^2 \delta(E_f - E_i)$$

The two (normalized) states that we use to solve for the matrix element V_{ba} are the state for particle type a, $\psi_a(x) = \sqrt{\frac{q}{2}}e^{-|q|x}$ and the state for a free particle (since it does not feel the effects of V_o) type b, $\psi_b(x) = \frac{e^{ikx}}{\sqrt{L}}$

$$\begin{aligned}
|V_{ba}| &= \langle\psi_b|V_{ab}|\phi_a\rangle \\
&= \int_{-\infty}^{\infty} \left(\frac{e^{-ikx}}{\sqrt{L}}\right)(\alpha)\left(\sqrt{\frac{q}{2}}e^{-|q|x}\right)\,dx \\
&= \frac{\alpha\sqrt{q}}{\sqrt{2L}}\int_{-\infty}^{\infty} e^{-ikx}e^{-|q|x}\,dx \\
&= \frac{\alpha\sqrt{q}}{\sqrt{2L}}\left(\int_{-\infty}^{0} e^{-ikx}e^{qx}\,dx + \int_{0}^{\infty} e^{-ikx}e^{-qx}\,dx\right) \\
&= \frac{\sqrt{q}\alpha}{\sqrt{2L}}\left|\frac{1}{q-ik}+\frac{1}{q+ik}\right| \\
&= \frac{\sqrt{q}\alpha}{\sqrt{2L}}\left|\frac{2q}{q^2+k^2}\right| \\
|V_{ba}|^2 &= \frac{q\alpha^2}{2L}\left(\frac{2q}{q^2+k^2}\right)^2 \\
&= \frac{2q^3\alpha^2}{L\left(q^2+k^2\right)^2}
\end{aligned}$$

The decay rate $\Gamma_{a\to b}$ is now given by

$$\begin{aligned}
\Gamma_{a\to b} &= \frac{\pi}{2\hbar}\int_{-\infty}^{\infty}\frac{L}{2\pi}|V_{ba}|^2\,\delta\left(E_k-(\hbar\omega-B)\right)\,dk \\
&= \frac{\pi}{2\hbar}\int_{-\infty}^{\infty}\frac{L}{2\pi}\left(\frac{2q^3\alpha^2}{L\left(q^2+k^2\right)^2}\right)\delta\left(E_k-(\hbar\omega-B)\right)\,\frac{dE}{dE/dk} \\
&= \frac{q^3\alpha^2}{2\hbar}\int_{-\infty}^{\infty}\left(\frac{1}{\left(q^2+k^2\right)^2}\right)\delta\left(E_k-(\hbar\omega-B)\right)\,\frac{m}{\hbar^2 k}dE \\
&= \frac{2mq^3\alpha^2}{\hbar^3 k(q^2+k^2)^2}
\end{aligned}$$

If we check our units we get

$$\begin{aligned}
\frac{2mq^3\alpha^2}{\hbar^3 k(q^2+k^2)^2} &= \frac{M\cdot(1/L)^3\cdot E^2}{(E\cdot T)^3\cdot(1/L)\cdot((1/L)^2+(1/L)^2)^2} \\
&= \frac{M\cdot(1/L)^3\cdot E^2}{(E\cdot T)^3\cdot(1/L)^5} \\
&= \frac{M}{E\cdot T^3\cdot(1/L)^2} \\
&= \frac{ML^2}{E\cdot T^3} \\
&= \frac{E}{E\cdot T} \\
&= \frac{1}{T}
\end{aligned}$$