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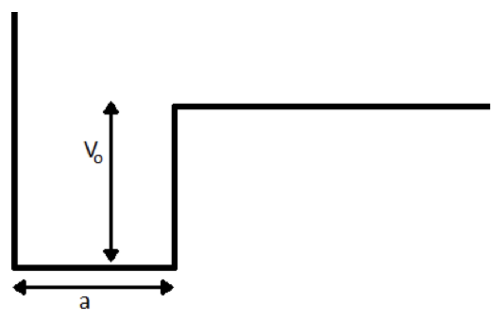
Chapter 2 Review Problems

Problem 1
Consider the one dimensional potential,

$$V(x) = \begin{cases} \infty & x < 0 \\ -V_o & 0 < x < a \\ 0 & x > a \end{cases}$$

1. For a fixed a find V_o for n bound states
2. At $t = 0$, the potential instantly disappears.For a particle originally in the ground state of the potential, what is the differential probability, dN/dp, for observing the particle with momentum p?



Part 1 :
There is one main assumption when solving for the bound states, that $E = 0$. This comes from the idea that this particle is not quite bound, $E < 0$, and not free either, $E > 0$. So we can start by writing the most general wave function.

$$\psi(x) = \begin{cases} 0 & x < 0 \\ sin(kx) & 0 < x < a \\ Ae^{-qx} & x > a \end{cases}$$

So, it can be seen from our assumption, that $q = \frac{2mE}{\hbar^2} = 0$, and so $\psi(x > a) = A$. Then, Using the Schroedinger equation, we can find an expression for V_o.

$$E = -V_o + \frac{\hbar^2 k^2}{2m} \Rightarrow V_o = \frac{\hbar^2 k^2}{2m}$$

Now, using the fact that $\psi(x > a)$ is a constant, we can say that,

$$\frac{d}{dx} sin(kx)\bigg|_a = kcos(ka) = 0$$

And so, for n bound states, ka is equal to odd integer multiples of $\frac{\pi}{2}$

$$ka = (2n-1)\frac{\pi}{2} \qquad \text{For} \quad m = 1, 2, 3...$$

Finally, plugging that into Eq. , we have the depth of the potential well for a given number of bound states.

$$V_o = \frac{\hbar^2\pi^2}{8ma^2}(2n-1)$$

Part 2 :

In general, the differential probability is given by the following,

$$\frac{dN}{dp} = \frac{1}{2\pi\hbar}|< p|\psi >|^2 = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dx < p|x >< x|\psi >= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dxe^{\frac{ipx}{\hbar}}\psi(x)$$

Then, using our equation for $\psi(x)$, we have the following.

$$\frac{dN}{dp} = \frac{1}{2\pi\hbar}\left(\int_0^a dxe^{\frac{ipx}{\hbar}}\sin(kx) + A\int_a^{\infty} dxe^{\frac{ipx}{\hbar}}e^{-qx}\right)$$

Note, we are ignoring the normalization constant, as its solution is trivial to find and would not contribute meaningfully to this solution. Which then can be simplified to,

$$\frac{dN}{dp} = \frac{1}{2\pi\hbar}\left(2i\int_0^a dxe^{\frac{ipx}{\hbar}}e^{ikx} - 2i\int_0^a dxe^{\frac{ipx}{\hbar}}e^{-ikx} + A\int_a^{\infty} dxe^{\frac{ipx}{\hbar}}e^{-qx}\right)$$

This then becomes very easy to integrate.

$$\frac{dN}{dp} = \frac{1}{\pi}\left(\frac{-1+e^{ai(p/\hbar-p)}}{\hbar k-p}\frac{-1+e^{ai(p/\hbar+p)}}{\hbar k+p}\right) + \frac{A}{2\pi}\left(\frac{-e^{a(q-ip/\hbar)}}{\hbar q-ip}\right)$$

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Problem 2
Consider a particle, mass m, under the influence of a potential

$$V(x) = V_0\Theta(-x) - \frac{\hbar^2}{2m}\beta\delta(x-a), \quad V_0 \to \infty,\ \beta > 0.$$

1. Find a trancendental equation for the energy of a bound state.
2. Consider now a plane wave incident on the potential from $x = \infty$ in the $-\hat{x}$ direction which is reflected off the potential. For $x > a$, the waveform is $e^{-ikx} - e^{2i\delta}e^{ikx}$. Find the phase shift of the reflected wave.

Part 1 :
First, determine the general form of the wavefunctions. Most generally,

$$\psi_I(x) = Ae^{qx} + Be^{-qx} \text{ and } \psi_{II}(x) = e^{-qx}.$$

Under the condition that $V_0 \to \infty$, the boundary condition $\psi_I(0) = 0$ must be met, thus $A + B = 0$. Applying this, we get that

$$\psi_I(x) = Ae^{qx} - Ae^{-qx} = C\sinh(qx)$$

for a new constant $C = 2A$.

Applying the boundary conditions:

1. $\psi_I(x)|_{x=a} = \psi_{II}(x)|_{x=a}$, thus: $C\sinh(qa) = e^{-qa}$.
2. $\frac{\partial}{\partial x}\psi_{II}(x)|_{x=a} - \frac{\partial}{\partial x}\psi_I(x)|_{x=a} = \beta\psi(x)$, thus: $-qe^{-qa} - qC\cosh(qa) = \beta e^{-qa}$.

Simplifying the second BC, we get $C\cosh(qa) = \left(\frac{-\beta}{q} - 1\right)e^{-qa} = \frac{-\beta-q}{q}e^{-qa}$. Then dividing the first BC by the second, we get:

$$\tanh(qa) = -\frac{q}{\beta + q}.$$

Recalling that $q = \sqrt{2mE/\hbar^2}$, we have found a transcendental equation for the energy of a bound state.

Part 2 :
First, we need to reconsider the wavefunctions. Most generally,

$$\psi_I(x) = A\sin(kx) + B\cos(kx) \text{ and } \psi_{II}(x) = e^{-ikx} - e^{2i\delta}e^{ikx} = \sin(kx + \delta).$$

Again, under the condition that $V_0 \to \infty$, the boundary condition $\psi_I(0) = 0$ must be met, thus $A\sin(0) + B\cos(0) = 0$, so $B = 0$. Then we can simplify the waveform:

$$\psi_{II}(x) = A\sin(kx).$$

Applying the boundary conditions:

1. $\psi_I(x)|_{x=a} = \psi_{II}(x)|_{x=a}$, thus: $A\sin(ka) = \sin(ka + \delta)$.
2. $\frac{\partial}{\partial x}\psi_{II}(x)|_{x=a} - \frac{\partial}{\partial x}\psi_I(x)|_{x=a} = \beta\psi(x)$, thus: $k\cos(ka - \delta) - kA\cos(ka) = \beta\sin(ka - \delta)$.

Simplifying the second BC, we get $A\cos(ka) = \cos(ka - \delta) - \frac{\beta}{k}\sin(ka - \delta)$. Then dividing the second BC by the first, we get:

$$\cot(ka) = \cot(ka - \delta) - \frac{\beta}{k},$$

and solving for the phase shift yields:

$$\delta = \cot^{-1}\left(\cot(ka) + \frac{\beta}{k}\right) - ka.$$

Problem 3
Here we will examine two problems dealing with a simple harmonic oscillator.

1. Calculate $\langle m|(a^\dagger a)^K a^\dagger (aa^\dagger)^M|n\rangle$ where $m = 1$ and $n = 0$.
2. In the case of the three dimensional case of a harmonic osciallator, given the quantum numbers n_x, n_y, and n_z, and that $N = n_x + n_y + n_z$, find the degeneracy in eigenstates up to $N = 2$.

Part 1 :
Here, we make use of the fact that the raising and lowering operators, respectively, act on eigenstates of the harmonic oscillator in the following way: $a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$ and $a\,|n+1\rangle = \sqrt{n+1}\,|n\rangle$.

Next, we can start with the $(aa^\dagger)^M$ operator. Ignoring the power of M, we can see that the raising operator raises the state with a coefficient of $\sqrt{n+1}$ and the lowering operator returns to the $|n\rangle$ state with a coefficient of $(n+1)$. Acting M times, we can simplify the expression to $(n+1)^M\langle m|(a^\dagger a)^K a^\dagger|n\rangle$. Then we apply the raising operator again to get $\sqrt{n+1}(n+1)^M\langle m|(a^\dagger a)^K|n+1\rangle$. Now we must apply the $(a^\dagger a)^K$ operator (also known as the number operator). We can apply this to the $|n+1\rangle$ state to get the form $(n+1)^{K+M}\sqrt{n+1}\,\langle m|n+1\rangle$.
Finally, given the states $m = 1$ and $n = 0$, our expression simplifies to $1^{K+M}\langle 1|1\rangle = 1$.

Part 2 :
In the one dimensional case, we know the energy to scale as $E_n = \hbar\omega(n + \frac{1}{2})$. In the three dimensional case, each dimension contributes its own similar term to the energy. Thus the energy becomes $E_N = \hbar\omega(n_x + \frac{1}{2}) + \hbar\omega(n_y + \frac{1}{2}) + \hbar\omega(n_z + \frac{1}{2}) = \hbar\omega(n_x + n_y + n_z + \frac{3}{2})$. If we substitute the given relation, we get $E_N = \hbar\omega(N + \frac{3}{2})$. Now, we must find the number of states that correspond to each of the first 3 energy levels.

In the ground state case, $N = 0$, there is obviously only one state where each quantum number is 0; so there is no degeneracy.

For $N = 1$, we need all the possible positive integers of n_x, n_y, and n_z that add to one. Since the numbers cannot be negative, we have three solutions:

$$n_x = 1, n_y = 0, n_z = 0$$
$$n_x = 0, n_y = 1, n_z = 0$$
$$n_x = 0, n_y = 0, n_z = 1$$

And thus, we have a degeneracy of three.

Finally in the $N = 2$ case, we again need to find each combination of integers that will result in $n_x + n_y + n_z = 2$. We can start by listing all the states with one dimension in the second excited state:

$$n_x = 2, n_y = 0, n_z = 0$$

$$n_x = 0, n_y = 2, n_z = 0$$

$$n_x = 0, n_y = 0, n_z = 2$$

Additionally, we must include both combinations of $n_x = 1$:

$$n_x = 1, n_y = 1, n_z = 0$$

$$n_x = 1, n_y = 0, n_z = 1$$

, and the remaining state in which $n_y = n_z = 1$:

$$n_x = 0, n_y = 1, n_z = 1$$

. All six of these states correspond to the same energy, meaning the degeneracy is 6 for $n + 2$.