PHY851 Final Preparation (Chapter 8)

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Problems (Scattering at Low Energies)

A beam of spinless particles of mass m and kinetic energy E is aimed as a spherically symmetric repulsive potential

$$V(r) = V_0 \Theta(a - r)$$

Assume $E < V_0$.

1. Find the $l = 0$ phase shift as a function of the incoming wave number k.
2. What is the cross section as $k \to 0$? What is the scattering length?
3. What is the relative probability for a particle in the wave packet to be at the origin compared to the probability with no potential? That is, if ρ_0 is the probability density at $r = 0$ in the absence of the potential and ρ is the density with the potential, find $\frac{\rho}{\rho_0}$.

Solutions

1. Using the definition $\psi(k, r) = rR_{l=0}(k, r)$, one can see the Schrodinger Equation looks exactly like the one-dimensional case. We have solutions of the following forms.

   In region I, where $r < a$, we have

$$\psi_I(r) \sim A \sinh(qr) \quad , \quad q = \frac{\sqrt{2m(V_0 - E)}}{\hbar} \tag{1}$$

   In region II, where $r > a$, we have a plane wave

$$\psi_{II}(r) \sim \sin(kr + \delta) \quad , \quad k = \frac{\sqrt{2m(E)}}{\hbar} \tag{2}$$

   Matching boundary conditions at $r = a$,

$$A \sinh(qa) = \sin(ka + \delta) \tag{3}$$

$$Aq \cosh(qa) = k \cos(ka + \delta) \tag{4}$$

   Dividing the top equation by the bottom, we see

$$\frac{1}{q} \tanh(qa) = \frac{1}{k} \tan(ka + \delta)$$

Rearranging,

$$\tan(ka+\delta) = \frac{k}{q}\tanh(qa)$$

$$\boxed{\delta = \tan^{-1}\left(\frac{k}{q}\tanh(qa)\right) - ka} \tag{5}$$

2. The total cross-section as a function of δ_l is given by:

$$\sigma = \frac{4\pi}{k^2}\sum(2l+1)\sin^2\delta_l. \tag{6}$$

In this case, we are looking at s-waves, so $l = 0$. At small k, we can rewrite our phase shift:

$$\delta_0(k) \approx \frac{k}{q}\tanh(qa) - ka \tag{7}$$

since

$$\tan^{-1} x \approx x$$

for small x.

Then our total cross-section becomes

$$\sigma \approx \frac{4\pi}{k^2}\delta_0^2 = \frac{4\pi}{k^2}\left(\frac{k}{q}\tanh(qa) - ka\right)^2$$

$$= 4\pi\left(\frac{1}{q}\tanh(qa) - a\right)^2$$

$$\sigma \approx 4\pi a^2\left(1 - \frac{\tanh(qa)}{qa}\right)^2 \tag{8}$$

as $k \to 0$. Note that letting $k \to 0$ is equivalent to letting the incident beam energy $E \to 0$.

The cross-section is dominated by the $l = 0$ contribution at low energy. To find the scattering length α, we recall its definition:

$$\alpha \equiv -\frac{\partial}{\partial k}\delta_0(k)|_{k=0}. \tag{9}$$

The form of the phase shift in Eq. (7) makes the derivative straightforward to calculate:

$$\boxed{\alpha = a - \frac{1}{q}\tanh(qa).} \tag{10}$$

Note that:

$$\alpha^2 = a^2\left(1 - \frac{\tanh(qa)}{qa}\right)^2$$

so we can express the cross-section (in Eq. (8)) in terms of the scattering length as:

$$\boxed{\sigma \approx 4\pi\alpha^2.} \tag{11}$$

If we let $V_0 \to \infty$, then $q \to \infty$, and the scattering length is the width of the well:

$$\alpha = a$$
$$\sigma = 4\pi a^2$$
$$\delta_0(k) \approx -ka.$$

The resulting cross-section is four times the classical hard-sphere scattering cross-section, πa^2.

3. Recalling equation (1) we can see that the probability density of observing a particle for $r < a$ with the potential is:

$$P = A^2 \frac{\sinh^2(qr)}{(kr)^2} \tag{12}$$

The probability density of observing a particle for $r < a$ without the potential is:

$$P_0 = \frac{\sin^2(kr)}{(kr)^2} \tag{13}$$

since the incoming wave must match the outgoing wave (equation (2), where $\delta = 0$ since $V_0 = 0$). Therefore, the relative probability for a particle in the wave packet to be observed at the origin compared to the probability without the potential is:

$$\begin{aligned} \frac{\rho}{\rho_0} &= A^2 \frac{\sinh^2(qr)}{\sin^2(kr)}\bigg|_{r=0} \\ &= A^2 \frac{q^2}{k^2} \end{aligned} \tag{14}$$

Which can be found using L'Hospital's rule. Next rearranging (3) we can express A as:

$$A = \frac{\sin(ka+\delta)}{\sinh(ka)} \tag{15}$$

Therefore our expression for the relative probability becomes:

$$\frac{\rho}{\rho_0} = \frac{q^2 \sin^2(ka+\delta)}{k^2 \sinh^2(ka)} \tag{16}$$

Using (5) and the trig identity, $\sin^2(\tan^{-1} x) = 1 - \frac{1}{1+x^2}$, one can show that:

$$\begin{aligned} \sin^2(ka+\delta) &= 1 - \frac{1}{1 + \frac{k^2}{q^2}\tanh^2(qa)} \\ &= \frac{k^2 \sinh^2(qa)}{q^2\cosh^2(qa) + k^2\sinh^2(qa)} \end{aligned} \tag{17}$$

inserting (17) back into (16) gives us our final answer:

$$\boxed{\frac{\rho}{\rho_0} = \frac{1}{\cosh^2(qa) + \frac{k^2}{q^2}\sinh^2(qa)}} \tag{18}$$