Adding Angular Momentum A neutron and proton occupy the ground state of a harmonic oscillator. The particles then feel two additional sources of interaction. First, they have a spin-spin interation,

$$V_{ss} = \alpha S_n \cdot S_p,$$

and secondly, they experience an external magnetic field

$$V_B = -(\mu_n S_n + \mu_p S_p) \cdot \vec{B}$$

Letting J and M reference the total angular momentum and its projection, and letting m_n and m_s reference the projections of the neutron and protons spins,

(a) (10 pts) Circle the operators that commute with the Hamiltonian

- The magnitude of the total angular momentum, $|\vec{J}^2| = \hbar^2 J(J+1)$
- J_z
- $S_z^{(n)}$
- $S_z^{(p)}$

(b) (10 pts) In the J, M basis,

$$|J=1, M=1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; |J=1, M=-1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$|J=1, M=0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \; |J=0, M=0\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Write the Hamiltonian as a 4×4 matrix.

(c) (5 pts) Find the eigen-energies of the Hamiltonian.

Solution.
(a) **Relevant Equations**

$$V_{ss} = \alpha S_n \cdot S_p$$

$$V_B = -(\mu_n S_n + \mu_p S_p) \cdot \vec{B}$$

Commutation Relations

$$[|J|^2, J_z] = 0$$
$$[|J|^2, |S_n|^2] = 0$$
$$[|J|^2, |S_p|^2] = 0$$
$$[|J|^2, S_z^{(n)}] \neq 0$$
$$[|J|^2, S_z^{(p)}] \neq 0$$
$$[J_z, S_z^{(n)}] = 0$$
$$[J_z, S_z^{(p)}] = 0$$
$$[S_z^{(n)}, S_z^{(p)}] = 0$$

Check $|J|^2$

$$[|J|^2, V_{ss}] = [|J|^2, |J|^2 - |S_n|^2 - |S_p|^2] = 0$$
$$[|J|^2, V_B] = [|J|^2, S_n + S_p] \neq 0$$

$|J|^2$ doesn't commute.

Check J_z

$$[J_z, V_{ss}] = [J_z, |J|^2 - |S_n|^2 - |S_p|^2] = 0$$
$$[J_z, V_B] = [J_z, S_n + S_p] = 0$$

J_z commutes.

Check $S_n^{(z)}$

$$[S_z^{(n)}, V_{ss}] = [S_z^{(n)}, |J|^2 - |S_n|^2 - |S_p|^2] \neq 0$$
$$[S_z^{(n)}, V_B] = [S_z^{(n)}, S_n + S_p] = 0$$

$S_z^{(n)}$ doesn't commute.

Check $S_p^{(z)}$

$$[S_z^{(p)}, V_{ss}] = [S_z^{(p)}, |J|^2 - |S_n|^2 - |S_p|^2] \neq 0$$
$$[S_z^{(p)}, V_B] = [S_z^{(p)}, S_n + S_p] = 0$$

$S_z^{(p)}$ doesn't commute.

Only J_z commutes with the full Hamiltonian.

(b) First, we have a spin-spin interaction between a neutron ($s_n = \frac{1}{2}$) and a proton ($s_p = \frac{1}{2}$) in the ground state of a harmonic oscillator. ($\ell = 0$) Since this interaction goes like $\vec{S}_n \cdot \vec{S}_p$, we know that there will only be J dependence, so V_{ss} will be diagonal in the J, M basis.

$$\vec{S}_n \cdot \vec{S}_p = \frac{1}{2}\left(\vec{J}^2 - \vec{S}_n^2 - \vec{S}_p^2\right)$$

$$\langle J, M| V_{ss} |J', M'\rangle = \delta_{J,J'}\delta_{M,M'}\frac{\alpha\hbar^2}{2}(J(J+1) - s_n(s_n+1) - s_p(s_p+1))$$

Since we are only dependent on J, we have

$$J = 1 \longrightarrow \frac{\alpha\hbar^2}{2}\left(2 - \frac{3}{2}\right) = \frac{\alpha\hbar^2}{4}$$

$$J = 0 \longrightarrow \frac{\alpha\hbar^2}{2}\left(0 - \frac{3}{2}\right) = -\frac{3\alpha\hbar^2}{4}$$

Thus, the 4×4 matrix, V_{ss} reads:

$$V_{ss} = \frac{\alpha\hbar^2}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

Next we have

$$V_B = -(\mu_n\vec{S}_n + \mu_p\vec{S}_p)\cdot\vec{B}.$$

Since V_b goes like $\vec{S}_n + \vec{S}_p$, we know that it will not be diagonal in the J, M basis, but rather in the m_n, m_p basis. So in order to write V_b, we must rewrite each of the J, M states in the m_n, m_p basis.

Start at the top, where $J, M = s_n + s_p = 1$, and $m_p = s_p, m_n = s_n$. We know there will only be one $|m_n, m_p\rangle$ state with $M = s_n + s_p$.

$$|J = 1, M = 1\rangle = |m_n = 1/2, m_p = 1/2\rangle$$

Similarly for $M = -1$, there will only be one corresponding $|m_n, m_p\rangle$ state,

$$|J = 1, M = -1\rangle = |m_n = -1/2, m_p = -1/2\rangle$$

Now applying the lowering operator to $|J = 1, M = 1\rangle$, we get coefficients for the state with the same $J = 1$, but $M = s_n + s_p - 1$.

$$J_- |J = 1, M = 1\rangle = \left(S_n^- + S_p^-\right)|m_n = 1/2, m_p = 1/2\rangle$$

$$|J = 1, M = 0\rangle = \frac{1}{\sqrt{2}}\left(|m_n = -1/2, m_p = 1/2\rangle + |m_n = 1/2, m_p = -1/2\rangle\right)$$

Since there are two $|m_n, m_p\rangle$ states corresponding to $|J = 1, M = 0\rangle$, there has to be another state in the J, M basis that corresponds with the same two $|m_n, m_p\rangle$ states, but orthogonal to the state above. We do this simply by making one of the coefficients negative.

$$|J = 0, M = 0\rangle = \frac{1}{\sqrt{2}}\left(|m_n = -1/2, m_p = 1/2\rangle - |m_n = 1/2, m_p = -1/2\rangle\right)$$

Note that the coefficients also could have been calculated using the fact that $J = |s_n \pm s_p|$ and $M = m_n + m_p \leq J$. The latter method is feasible for this problem since the system is in the ground state of a harmonic oscillator ($\ell = 0$), giving only four total states. For a system with larger ℓ, it would be best to use the raising and lowering operators to calculate coefficients.

In the m_n, m_p basis, the matrix elements for V_b go like

$$\langle m_p, m_n | V_B | m_p', m_n' \rangle = -\delta_{m_p,m_p'}\delta_{m_n,m_n'} B\hbar(\mu_p m_p + \mu_n m_n)$$

Now, calculate the diagonal elements in the new basis:

$$\langle J=1, M=1 | V_B | J=1, M=1 \rangle = \frac{-B\hbar}{2}(\mu_p + \mu_n)$$

$$\langle J=1, M=-1 | V_B | J=1, M=-1 \rangle = \frac{B\hbar}{2}(\mu_p + \mu_n)$$

It can be seen that

$$\langle J=1, M=0 | V_B | J=1, M=0 \rangle = 0$$

Similarly for

$$\langle J=0, M=0 | V_B | J=0, M=0 \rangle = 0$$

We need to look at the off diagonal elements. Note that there will only be overlap when $M = M'$.

$$\langle J=1, M=0 | V_B | J=0, M=0 \rangle = \frac{B\hbar}{2}(\mu_p - \mu_n)$$

$$\langle J=0, M=0 | V_B | J=1, M=0 \rangle = -\frac{B\hbar}{2}(\mu_p - \mu_n)$$

Putting it all together

$$V_B = -\frac{B\hbar}{2}\begin{pmatrix} (\mu_p + \mu_n) & 0 & 0 & 0 \\ 0 & (-\mu_p - \mu_n) & 0 & 0 \\ 0 & 0 & 0 & (\mu_p - \mu_n) \\ 0 & 0 & (\mu_p - \mu_n) & 0 \end{pmatrix}$$

Now we can write our complete 4×4 Hamiltonian,

$$H = V_{ss} + V_B = \begin{pmatrix} \frac{\alpha\hbar^2}{4} - \frac{\hbar B}{2}(\mu_n + \mu_p) & 0 & 0 & 0 \\ 0 & \frac{\alpha\hbar^2}{4} + \frac{\hbar B}{2}(\mu_n + \mu_p) & 0 & 0 \\ 0 & 0 & \frac{\alpha\hbar^2}{4} & -\frac{\hbar B}{2}(\mu_p - \mu_n) \\ 0 & 0 & -\frac{\hbar B}{2}(\mu_p - \mu_n) & -\frac{3\alpha\hbar^2}{4} \end{pmatrix}$$

(c) We can read off the first two eigen-energies since they are diagonal in the Hamiltonian.

$$\epsilon_1 = \frac{\alpha\hbar^2}{4} - \frac{\hbar B}{2}(\mu_n + \mu_p)$$

$$\epsilon_2 = \frac{\alpha\hbar^2}{4} + \frac{\hbar B}{2}\left(\mu_n + \mu_p\right)$$

We can calculate the remaining two eigen-energies by looking at the sub-2×2 matrix and writing it in terms of Pauli sigma matrices.

$$H_{2\times 2} = -\frac{\alpha\hbar^2}{4}\mathbb{I} + \frac{\alpha\hbar^2}{2}\sigma_z - \frac{\hbar B}{2}\left(\mu_p - \mu_n\right)\sigma_x$$

Now we get the final two eigen-energies.

$$\epsilon_{3,4} = -\frac{\alpha\hbar^2}{4} \pm \sqrt{\left(\frac{\hbar^2\alpha}{2}\right)^2 + \left(\frac{\hbar B}{2}\right)^2\left(\mu_p - \mu_n\right)^2}$$