# PHY851 Chapter 1 Review: Time Evolution of Two-State Systems

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**Introduction: Time-Evolution of Two-State System Problems**
These problems consist of a particle with two possible states and some time-dependent probability of changing from state A to state B. Common scenarios include:

- Neutrino oscillation from one flavor to another (Cp 1, Ex 1.6 and Problem 17)
- Neutral Kaon oscillation (Cp 1, Problem 16; In-Class Exercise 2)
- $\frac{1}{2}$-spin systems evolving between spin-up and spin-down (Cp 1, Ex. 1.5; Exam 1 Problem 1)

Possible questions include:

- Find the energy of the particle in state A versus state B given the Hamiltonian.
- Given the particle starts in state A, find the probability that the particle is in state A or B as a function of time.
- Given the particle starts in state A, find the probability that the particle is in state A or B as a function of the distance traveled by the particle.
- Bonus: Particle could decay at a different rate in state A versus state B.

**General approach to solving these problems:**

1. Identify the free term and the mixing term of the Hamiltonian. The mixing term's matrix form will have off-diagonal elements. The Hamiltonian will be: $H = H_{\text{free}} + H_{\text{mix}}$.
2. Expand the time-evolution operator, $U(t) = e^{-iHt/\hbar}$, into matrix form. The Baker-Campbell-Hausdorff relation may be used if the correct commutator conditions for $H_{\text{free}}$ and $H_{\text{mix}}$ are met. If the Hamiltonian can be decomposed into $\sigma_x, \sigma_y, \sigma_z$ components, the identity for Pauli matrix exponentials may be useful: $e^{i\theta\hat{\sigma}\cdot\hat{n}} = \cos(\theta) + i\hat{\sigma}\cdot\hat{n}\sin(\theta)$
3. To find energy of eigenstates: find eigenvalues of the Hamiltonian
4. To find time-dependent probability of particle being in target state B given starting state A, use: $P_B(t) = |\langle B|\, U(t)\, |A\rangle|^2$
5. To find probability in terms of distance traveled by particle, find $P_B(t)$ and make the substitution $t = \frac{x}{\gamma v}$

**Side note about units and relativity:** There are two common unit systems for describing particle interactions.

*Unit system version 1, Planck Units (i.e "God's Units"):*
Take $c = \hbar = 1$, unitless. Then time is measured in units of distance, and mass and energy are in units of inverse distance. Then $v$ is unitless and:

$$\gamma = \frac{1}{\sqrt{1-v^2}} = \frac{E}{m}; \qquad t = \frac{x}{\gamma v}$$

where $E$ is total energy (kinetic + rest), $t$ in the particle frame, $v, x$ in the lab frame.

*Unit system version 2, human units:*
Take $c = \text{m/s}$, $\hbar = \text{MeV} \cdot \text{s}$. Then mass is given as $\text{MeV}/c^2$:

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{E}{mc^2}; \qquad t = \frac{x}{\gamma v}$$

where $E$ is total energy (kinetic + rest), $t$ in the particle frame, $v, x$ in the lab frame.

If $v = ac$ (i.e. velocity is given as a fraction of the speed of light), then:

$$t = \frac{x}{\gamma a c}$$

This is a convenient format to use when finding a numerical solution because we will end up with something of the form $\frac{t}{\hbar} = \frac{x}{\gamma a c \hbar}$ and $c\hbar$ is a common numerical constant to have handy.

**Example Problem: Neutral Kaon Oscillations**
There are two kinds of neutral kaons one can make using down and strange quarks:

$$\left|K^0\right\rangle = \left|d\bar{s}\right\rangle, \left|\bar{K}^0\right\rangle = \left|s\bar{d}\right\rangle$$

If it were not for the weak interaction, the two species would have equal masses and the Hamiltonian for a Kaon with zero momentum would be:

$$H_0 = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}$$

There is an additional term from the weak interaction that mixes the states:

$$H_{\text{mix}} = \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix}$$

The masses of a neutral kaon are 497.6 $\text{MeV}/c^2$, without mixing, but after adding the missing term the masses differ by $3.56\mu\text{eV}/c^2$. The two eigenstates are known as K-short ($K_S$) and K-long ($K_L$) because they decay with quite different lifetimes.

**Problem 1**
What is $\varepsilon$?

We find $\varepsilon$ by taking the eigenvalues of the Hamiltonian:

$$|H_0 + H_m| = \begin{vmatrix} M - \lambda & \varepsilon \\ \varepsilon & M - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 2M\lambda + (M^2 -^2) = 0$$

$$\lambda = \frac{2M \pm \sqrt{4M^2 - 4(M^2 - \epsilon^2)}}{2} = M \pm \varepsilon$$

So the difference between the two states is $2\varepsilon$, and:

$$\boxed{\varepsilon = \frac{\Delta mc^2}{2} = \frac{3.56}{2}\ \mu\text{eV} = 1.78\ \mu\text{eV}}$$

Where $\Delta mc^2$ is the difference in rest energy between the two states, which is the same as the difference in mass if mass is given in energy units. Note that the Hamiltonian is in units of energy so $\varepsilon$ has to be in units of energy.

**Problem 2**
If one creates a kaon in the $|K_0\rangle$ state at time t = 0, find the probability it would be measured as a $|\bar{K}_0\rangle$ as a function of time.

The time evolution operator is given by $U(t) = e^{-iHt/\hbar}$. Note the use of the Baker-Campbell-Hausdorff theorem here is trivial since $[H_0, H_m] = 0$:

$$H = H_0 + H_m = M\mathbb{I} + \varepsilon\sigma_x; \qquad [H_0, H_m] = M\varepsilon[\mathbb{I}, \sigma_x] = 0$$

$$U(t) = e^{-iHt/\hbar} = e^{-i(M\mathbb{I}+\varepsilon\sigma_x)t} = e^{-iMt/\hbar}e^{-i\varepsilon\sigma_x t/\hbar}\cancelto{1}{e^{-iM\varepsilon[\mathbb{I},\sigma_x]t/\hbar}} = e^{-iMt/\hbar}e^{-i\varepsilon\sigma_x t/\hbar}$$

Now we can use this identity to expand exponentials of Pauli matrices:

$$e^{i\theta\hat{\sigma}\cdot\hat{n}} = \cos\theta + i\hat{\sigma}\cdot\hat{n}\sin\theta$$

Putting it all together:

$$\begin{aligned}
|\psi(t)\rangle &= U(t)\,|K_0\rangle \\
&= e^{-iHt/\hbar}\,|K_0\rangle \\
&= e^{-iMt/\hbar}e^{-i\varepsilon\sigma_x t/\hbar}\,|K_0\rangle \\
&= e^{-iMt/\hbar}(\cos(\varepsilon t/\hbar) - i\sigma_x\sin(\varepsilon t/\hbar))\begin{pmatrix}1\\0\end{pmatrix} \\
&= e^{-iMt/\hbar}\begin{pmatrix}\cos(\varepsilon t/\hbar) & -i\sin(\varepsilon t/\hbar)\\ -i\sin(\varepsilon t/\hbar) & \cos(\varepsilon t/\hbar)\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \\
&= e^{-iMt/\hbar}\begin{pmatrix}\cos(\varepsilon t/\hbar)\\ -i\sin(\varepsilon t/\hbar)\end{pmatrix}
\end{aligned}$$

Then $P_{\bar{K}_0}(t)$ for this state is: $|\langle\bar{K}_0|\psi(t)\rangle|^2$:

$$\begin{aligned}
P_{\bar{K}_0}(t) &= |\langle\bar{K}_0|\,U\,|K_0\rangle|^2 \\
&= |\langle\bar{K}_0|\psi(t)\rangle|^2 \\
&= \left|e^{-iMt/\hbar}\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}\cos(\varepsilon t/\hbar)\\ -i\sin(\varepsilon t/\hbar)\end{pmatrix}\right|^2 \\
&= e^{-iMt/\hbar}e^{iMt/\hbar}(-i\sin(\varepsilon t/\hbar))(i\sin(\varepsilon t/\hbar)) \\
&= \sin^2\left(\frac{\varepsilon t}{\hbar}\right)
\end{aligned}$$

So the probability as a function of time is:

$$\boxed{P_{\bar{K}_0}(t) = \sin^2\left(\frac{\varepsilon t}{\hbar}\right)}$$

**Problem 3**
A beam of kaons is created in the $K_0$ channel and has a kinetic energy of 600 MeV per kaon. Plot the probability that the kaon is in the $K_0$ state as a function of the distance traveled, $x$. Ignore the fact that the kaons decay.

Via analogous work to Part 2, we find that the probability that the kaon is in the $K_0$ state is:

$$P_{K_0}(t) = \cos^2\left(\frac{\varepsilon t}{\hbar}\right)$$

To get the probability in terms of distance traveled instead of time, make the substitution:

$$t = \frac{x}{\gamma v}$$

where $v$ and $x$ are in the lab frame.

$$\Rightarrow P_{K_0}(x) = \cos^2\left(\frac{\varepsilon x}{\gamma v \hbar}\right)$$

Possible exam problem: Determine $x$ for which $P_{K_0} = 0$ and where $P_{K_0} = 1$.

$$P_{K_0} = 1 \text{ when } \frac{\varepsilon x}{\gamma v \hbar} = n\pi \Rightarrow x = \frac{n\pi\gamma v\hbar}{\varepsilon}$$
$$P_{K_0} = 0 \text{ when } \frac{\varepsilon x}{\gamma v \hbar} = \frac{(2n+1)\pi}{2} \Rightarrow x = \frac{(2n+1)\pi\gamma v\hbar}{2\varepsilon}$$

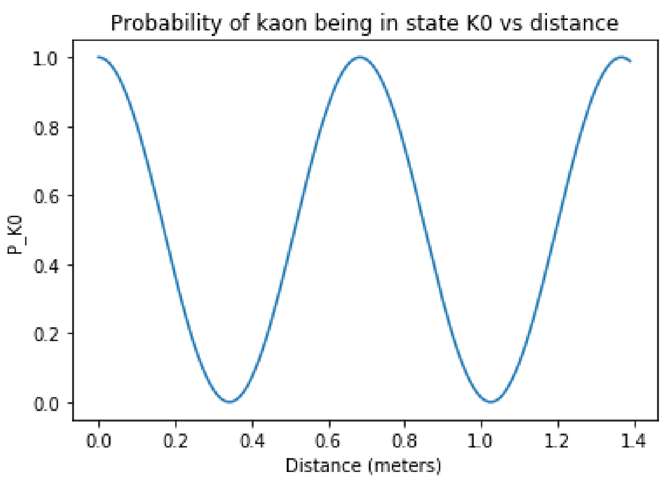
Or, if given enough information, we could find the numerical values for these elements:

$$E = E_{\text{rest}} + E_{\text{kinetic}} = mc^2 + E_{\text{kinetic}} = 497.6 + 600 = 1097.6 \text{ MeV}$$
$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \to \frac{v}{c} = \frac{\sqrt{\gamma^2 - 1}}{\gamma} = 0.89 \to v = 0.89c$$
$$\varepsilon = 1.78 \times 10^{-12} \text{ MeV from Part 1}$$
$$\hbar c = 1.97 \times 10^{-13} \text{ MeV} \cdot \text{m}$$
$$\Rightarrow \frac{\epsilon}{\gamma v \hbar} = \frac{\epsilon}{\gamma(0.89c)\hbar} = 4.596 \text{ m}^{-1}$$

So our probability in terms of $x$ is:

$$\boxed{P_{K_0}(x) = \cos^2(4.59x)}$$

The plot of $P_{K_0}(x)$ versus distance shows a cosine-squared wave with period: $\frac{\pi}{4.59} = 0.68$ meters.



**Problem 4**
Repeat (c), but take into account the decays.

Note that:

$$|K_S\rangle = \frac{1}{\sqrt{2}}(|K_0\rangle + |\bar{K}_0\rangle)$$
$$|K_L\rangle = \frac{1}{\sqrt{2}}(|K_0\rangle - |\bar{K}_0\rangle)$$

And that the decay term goes like $e^{-t/2\tau}$.

First, rewrite $|K_0\rangle$ as follows so that we can easily pull out the $|K_S\rangle$ and $|K_L\rangle$ states:

$$|K_0\rangle = \begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}\left[\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}1\\-1\end{pmatrix}\right]$$

Note that deriving the time-dependent wave function requires a bit more effort than previously, specifically in evaluating the mixing term:

$$[\cos(\varepsilon t/\hbar) - i\sigma_x \sin(\varepsilon t/\hbar)]\left[\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}1\\-1\end{pmatrix}\right]$$

Considering the effects of the original Hamiltonian and including the factors for the decay times $\tau_S$ and $\tau_L$, the time-dependent wave function becomes:

$$|\psi(t)\rangle = U(t)\,|K_0\rangle = \frac{1}{2}e^{-iMt/\hbar}\left[\begin{pmatrix}1\\1\end{pmatrix}e^{-i\varepsilon t/\hbar}e^{-t/2\tau_S} + \begin{pmatrix}1\\-1\end{pmatrix}e^{i\varepsilon t/\hbar}e^{-t/2\tau_L}\right]$$

Now we take $P_{K_0}$:

$$\begin{aligned}
P_{K_0}(t)) &= |\,\langle K_0|\,U\,|K_0\rangle\,|^2 \\
&= |\,\langle K_0|\psi(t)\rangle\,|^2 \\
&= \left|e^{-iMt/\hbar}\begin{pmatrix}1 & 0\end{pmatrix}\frac{1}{2}\left[\begin{pmatrix}1\\1\end{pmatrix}e^{-i\varepsilon t/\hbar}e^{-t/2\tau_S} + \begin{pmatrix}1\\-1\end{pmatrix}e^{i\varepsilon t/\hbar}e^{-t/2\tau_L}\right]\right|^2 \\
&= \frac{1}{4}\left|e^{-iMt/\hbar}(e^{-i\varepsilon t/\hbar - t/2\tau_S} + e^{i\varepsilon t/\hbar - t/2\tau_L})\right|^2 \\
&= \frac{1}{4}e^{-iMt/\hbar}e^{iMt/\hbar}\left|(e^{-i\varepsilon t/\hbar - t/2\tau_S} + e^{i\varepsilon t/\hbar - t/2\tau_L})\right|^2 \\
&= \frac{1}{4}\left|(e^{-i\varepsilon t/\hbar - t/2\tau_S} + e^{i\varepsilon t/\hbar - t/2\tau_L})\right|^2
\end{aligned}$$

Now substitute in $t = \frac{x}{\gamma v}$, where $x$ and $v$ are in the lab frame and $t$ is the time in the particle frame:

$$\boxed{P_{K_0}(x) = \frac{1}{4}\left|e^{-i\varepsilon x/(\gamma v\hbar) - x/(2\gamma v\tau_S)} + e^{i\varepsilon x/(\gamma v\hbar) - x/(2\gamma v\tau_L)}\right|^2}$$

Or, if we fully expand the state, we get:

$$P_{K_0} = \frac{1}{4}\left(e^{-x/(\gamma v\tau_S)} + e^{-x/(\gamma v\tau_L)} + e^{-x/(2\gamma v\tau_S) - x/(2\gamma v\tau_L)}\left(e^{-2i\varepsilon x/(\gamma v\hbar)} + e^{2i\varepsilon x/(\gamma v\hbar)}\right)\right)$$

The $K_L$ state dominates over the $K_S$ state, so this quickly becomes:

$$\begin{aligned}
P_{K_0}(x) &\simeq \frac{1}{4}\left|\cancelto{0}{e^{-i\varepsilon x/(\gamma v\hbar) - x/(2\gamma v\tau_S)}} + e^{i\varepsilon x/(\gamma v\hbar) - x/(2\gamma v\tau_L)}\right|^2 \\
&\simeq \frac{1}{4}\left|e^{i\varepsilon x/(\gamma v\hbar) - x/(2\gamma v\tau_L)}\right|^2 \\
&\simeq \frac{1}{4}e^{-x/(\gamma v\tau_L)}
\end{aligned}$$