Chapter 6: Approximation Methods

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1 Common Test Problems

- Stationary-State Perturbation Theory (SSPT) was most common, then Variational Theory
- One problem used the sudden approximation
- Maybe WKB is next?

2 Stationary-State Perturbation Theory

With this, you can solve for corrections to states and energies after a perturbation is applied.

2.1 General Problem/Strategy

Given some initial Hamiltonian H_0, with a perturbation V,

- Find the first-order correction for the energy

$$E_n^{(1)} = \langle n| V |n\rangle$$

- Find the second-order correction to the energy

$$E_n^{(2)} = -\sum_{m\neq n} \frac{|\langle m| V |n\rangle|^2}{\epsilon_m - \epsilon_n}$$

- Find the first-order correction to the wave function

$$\left|N^{(1)}\right\rangle = -\sum_{m\neq n} |m\rangle \frac{\langle m| V |n\rangle}{\epsilon_m - \epsilon_n}$$

2.2 Common Problem: Harmonic Oscillator

$$H_0 = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 x^2$$
$$V = V(X, P)$$

V isn't necessarily a function of X and P, but it's likely that it will be.

- Convert P, X into their raising and lowering operator form

$$X = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)$$

$$P = -i\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger)$$

- Remember that $a|0\rangle = 0$ because you can't lower it further than the ground state
- Usually, most of the terms in the sums will go to zero because the potential won't cause them to overlap

Practice Fall Final 2019 Problem 5:
Consider a particle of mass m in a one-dimensional harmonic oscillator potential with fundamental frequency ω,

$$H_0 = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

To second order in perturbation theory, what is the correction to the ground state energy when the perturbation

$$V = \beta P,$$

is added to the system.
Solution: Converting P into it's raising and lowering operator form, we get

$$P = -i\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger) = i\sqrt{\frac{\hbar m\omega}{2}}(a^\dagger - a).$$

The second-order energy correction is

$$E_n^{(2)} = -\sum_{m\neq n} \frac{|\langle m| V |n\rangle|^2}{\epsilon_m - \epsilon_n},$$

where $n = 0$ and $\epsilon_n = \frac{\hbar\omega}{2}$ because it's in the ground state. Plugging in the potential, we find that

$$V|0\rangle = \beta P|0\rangle = \beta i\sqrt{\frac{\hbar m\omega}{2}}(a^\dagger - a)|0\rangle .$$

The lowering operator results in a zero, leaving only the raising operator, so we get

$$V|0\rangle = \beta i\sqrt{\frac{\hbar m\omega}{2}}\sqrt{0+1}\,|1\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\,|1\rangle$$

By inspection, we can see that the sum for the energy correction will vanish unless $m = 1$, so we get

$$E_n^{(2)} = -\beta^2\frac{\hbar m\omega}{2}\frac{|\langle 1|1\rangle|^2}{\frac{3}{2}\hbar\omega - \frac{\hbar\omega}{2}}.$$

After simplifying, this becomes

$$E_n^{(2)} = -\beta^2\frac{\hbar^2\omega^2 m}{2\hbar\omega} = -\frac{\beta^2 m}{2}.$$

3 More Practice Problems

- For more SSPT problems, look to Subject Exam August 2021 problem 1, and HW problems 6.5, 6.6, 6.7
- For Variational Theory problems, look to Final Fall 2019 problem 6, Final Fall 2020 problem 6, and HW problems 6.3 and 6.4
- For Sudden Approximation problems, look to Midterm 1 Fall 2020 problem 3

4 Harmonic Oscillator with Exponential Perturbation

1. A particle is in the ground state of the harmonic oscillator, i.e. a Hamiltonian

   $$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

   with a perturbative potential

   $$V(x) = V_0 e^{\frac{x}{x_0}}.$$

   (a) Find the first order correction to the energy.

   $$V(x) = V_0 e^{\frac{x}{x_0}} = V_0 e^{\beta(a^\dagger + a)} = V_0 e^{\frac{\beta^2}{2}} e^{\beta a^\dagger} e^{\beta a}$$

   $$E^{(1)} = \langle 0|V|0\rangle = V_0 e^{\frac{\beta^2}{2}}\langle 0|e^{\beta a^\dagger}e^{\beta a}|0\rangle = V_0 e^{\frac{\beta^2}{2}}$$

Since $e^a = 1 + O(a^1)$ acting on the right only produces a nonzero result for only the first term in the series. Similar for $e^{a^\dagger}$ acting on the left.

(b) Calculate the matrix element $\langle n|V|0\rangle$ for $n = 1, 2, \ldots$

$$\langle n|V|0\rangle = V_0 e^{\frac{\beta^2}{2}} \langle n|e^{\beta a^\dagger} e^{\beta a}|0\rangle = V_0 e^{\frac{\beta^2}{2}} \langle n|e^{\beta a^\dagger}|0\rangle = V_0 e^{\frac{\beta^2}{2}} \langle n| \sum_m \frac{\beta^m (a^\dagger)^m}{m!}|0\rangle$$

$$= V_0 e^{\frac{\beta^2}{2}} \langle n| \sum_m \frac{\beta^m}{\sqrt{m!}}|m\rangle = V_0 e^{\frac{\beta^2}{2}} \frac{\beta^n}{\sqrt{n!}}$$

(c) Find the first order correction to the wave function.

$$\varepsilon_n - \varepsilon_0 = n\hbar\omega$$

$$|0^{(1)}\rangle = -\sum_n \frac{\langle n|V|0\rangle}{\varepsilon_n - \varepsilon_0}|n\rangle = -\frac{V_0 e^{\frac{\beta^2}{2}}}{\hbar\omega} \sum_n \frac{\beta^n}{n\sqrt{n!}}|n\rangle$$

(d) Find the second order correction to the energy.

$$E^{(2)} = -\sum_n \frac{|\langle n|V|0\rangle|^2}{\varepsilon_n - \varepsilon_0} = -\frac{V_0^2 e^{\beta^2}}{\hbar\omega} \sum_n \frac{(\beta^2)^n}{n(n!)}$$