Chapter 7 Problem: Differential Solid Angle and Structure Function

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Problem Suppose that the differential scattering angle $\frac{d\sigma}{d\Omega}$ for a point-like scatterer is α, independent of scattering angle. Now suppose we have $N^3 \gg 1$ of these scatterers arranged in a 3-dimensional cubic lattice with spacing a and side length Na. Consider a high energy plane wave incident with wavevector $\vec{k_i} = k\hat{z}$. Define $\vec{q} \equiv \vec{k_i} - \vec{k_f} = k\left[(1-\cos\theta_s)\,\hat{z} - \sin\theta_s\cos\phi_s\hat{x} - \sin\theta_s\sin\phi_s\hat{y}\right]$ where θ_s and ϕ_s are the scattering angles.

a) What is $\frac{d\sigma}{d\Omega}$ in terms of α, N, k, a, and $\vec{q}$?

b) Find the condition for $\vec{q}$ that *maximizes* the differential cross section.

c) In the limit that $N \to \infty$, find the condition for $\vec{q}$ that *minimizes* $\frac{d\sigma}{d\Omega}$.

d) Repeat parts **a–c** but with a lattice spacing of a (and side length of Na) in the $\hat{x}$ and $\hat{y}$ directions and a spacing of $2a$ (and side length of $2Na$) in the $\hat{z}$ direction.

Relevant Equations

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{point}} \tilde{S}(\vec{q}) \tag{1}$$

$$\tilde{S}(\vec{q}) = \frac{1}{N}\left|\sum_{\vec{a}} e^{i\vec{q}\cdot\vec{a}}\right|^2 \tag{2}$$

Solution

a) We use Eq. 2 to find the structure function,

$$\begin{aligned}
\sum_{\vec{a}} e^{i\vec{q}\cdot\vec{a}} &= \sum_{n_x,n_y,n_z=0}^{N} e^{i\vec{q}\cdot a(n_x\hat{x}+n_y\hat{y}+n_z\hat{z})} \\
&= \sum_{n_x,n_y,n_z=0}^{N} e^{iq_x a n_x} e^{iq_y a n_y} e^{iq_z a n_z} \\
&= \prod_{\hat{u}\in\{\hat{x},\hat{y},\hat{z}\}} \sum_{n=0}^{N} \left(e^{iq_u a}\right)^n \\
\implies \tilde{S}(\vec{q}) &= \frac{1}{N^3} \left| \prod_{\hat{u}\in\{\hat{x},\hat{y},\hat{z}\}} \sum_{n=0}^{N} \left(e^{iq_u a}\right)^n \right|^2
\end{aligned}$$

Then we plug this into Eq. 1.

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha}{N^3} \left| \prod_{\hat{u}\in\{\hat{x},\hat{y},\hat{z}\}} \sum_{n=0}^{N} \left(e^{iq_u a}\right)^n \right|^2}$$

b) To maximize $\frac{d\sigma}{d\Omega}$ we note that the maximum value of each term in the summations is 1, which occurs when $q_u a = 2j\pi$ where $j \in \mathbb{Z}$. So, the maximum occurs when this is true for each $\hat{u}$,

$$\boxed{\begin{cases} q_x a = 2r\pi \\ q_y a = 2s\pi \text{ where } r, s, t \in \mathbb{Z} \\ q_z a = 2t\pi \end{cases}}$$

c) To minimize $\frac{d\sigma}{d\Omega}$ we note that each individual term in the summations must have $\left|\left(e^{iq_u a}\right)^n\right|^2 = 1$. But, the phases of each term will only align when $e^{iq_u a} = 1$, otherwise the $(\cdot)^n$ will rotate the phase by $q_u a$ for each successive n. This heuristic explanation shows that the sum will always be bounded (but may not necessarily converge) so long as $e^{iq_u a} \neq 1$ for some q_u. Therefore, as $N \to \infty$, $\frac{d\sigma}{d\Omega} \to 0$ when

$$\boxed{\begin{cases} q_x a \neq 2r\pi \\ q_y a \neq 2s\pi \text{ for all } r, s, t \in \mathbb{Z} \\ q_z a \neq 2t\pi \end{cases}}$$

d) It is not necessary to work the entire problem again. We can see that replacing the spacing on the $\hat{z}$ axis is equivalent to replacing n_z with $2n_z$, and as such all q_z are replaced with $2q_z$ in the preceding parts.

$$\boxed{\begin{gathered}\frac{d\sigma}{d\Omega} = \frac{\alpha}{N^3}\left|\sum_{n=0}^{N}\left(e^{iq_x a}\right)^n \sum_{n=0}^{N}\left(e^{iq_y a}\right)^n \sum_{n=0}^{N}\left(e^{2iq_z a}\right)^n\right|^2 \\ \text{Maximized when } \begin{cases} q_x a = 2r\pi \\ q_y a = 2s\pi \text{ where } r,s,t \in \mathbb{Z} \\ q_z a = t\pi \end{cases} \\ \text{Minimized when } \begin{cases} q_x a \neq 2r\pi \\ q_y a \neq 2s\pi \text{ for all } r,s,t \in \mathbb{Z} \\ q_z a \neq t\pi \end{cases}\end{gathered}}$$