Problem 4.8a. Assuming that the Hamiltonian is invariant under time reversal, prove that the wave function for a spinless nondegenerate system at any given instant of time can always be chosen to be real.

Call $|n\rangle$ the eigenstates of H with energy E_n. Then using that H is invariant under Θ,

$$H\Theta|n\rangle = \Theta H|n\rangle = E_n\Theta|n\rangle\,,$$

meaning that the energy of $\Theta|n\rangle$ is E_n. We assumed that the system is nondegenerate, so this means $\Theta|n\rangle$ and $|n\rangle$ are the same state, differing by at most a phase independent of $\mathbf{r}$. If $\Theta|n\rangle = e^{i\delta}|n\rangle$, we can redefine the energy eigenstates $|\tilde{n}\rangle \equiv e^{i\delta/2}|n\rangle$ so that

$$\begin{aligned}\Theta|\tilde{n}\rangle &= \Theta(e^{i\delta/2}|n\rangle)\\ &= e^{-i\delta/2}e^{i\delta}|n\rangle\\ &= |\tilde{n}\rangle\,.\end{aligned}$$

Recall that

$$\begin{aligned}\Theta\psi_{\tilde{n}}(\mathbf{r}) &= \psi^*_{\tilde{n}}(\mathbf{r})\\ \Rightarrow \Theta\langle\mathbf{r}|\tilde{n}\rangle &= \langle\mathbf{r}|\tilde{n}\rangle^*\,,\end{aligned}$$

but we just showed that

$$\Theta\langle\mathbf{r}|\tilde{n}\rangle = \langle\mathbf{r}|\tilde{n}\rangle\,,$$

so

$$\langle\mathbf{r}|\tilde{n}\rangle = \langle\mathbf{r}|\tilde{n}\rangle^*\,,$$

therefore using this phase convention $\psi_{\tilde{n}}(\mathbf{r})$ is real.

Problem 4.8b. The wave function for a plane-wave state at $t = 0$ is given by a complex function $e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}$. Why does this not violate time-reversal invariance?

The statement in (**a**) doesn't apply here because $e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}$ is a degenerate state with $e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar}$.

Problem 4.9. Let $\phi(\mathbf{p})$ be the momentum-space wave function for state $|\alpha\rangle$, that is, $\phi(\mathbf{p}) = \langle\mathbf{p}|\alpha\rangle$. Is the momentum-space wave function for the time-reversed state $\Theta|\alpha\rangle$ given by $\phi(\mathbf{p})$, $\phi(-\mathbf{p})$, $\phi^*(\mathbf{p})$, or $\phi^*(-\mathbf{p})$? Justify your answer.

First a few background tidbits: From class notes, we know $\Theta\mathbf{P}\Theta^{-1} = -\mathbf{P}$, in other words, $\mathbf{P}$ is odd under time reversal. When we apply this to a state,

$$\mathbf{P}\Theta|\mathbf{p}\rangle = -\Theta\mathbf{P}\Theta^{-1}\Theta|\mathbf{p}\rangle = (-\mathbf{p})\Theta|\mathbf{p}\rangle$$

We see that $\Theta|\mathbf{p}\rangle$ is the momentum eigenstate corresponding to eigenvalue $-\mathbf{p}$. Now on to solving the problem at hand:

We express α as

$$|\alpha\rangle = \int \langle\mathbf{p}|\alpha\rangle^*|\mathbf{p}\rangle\,d^3p.$$

So then let us write $\Theta |\alpha\rangle$ as

$$\begin{aligned}\Theta |\alpha\rangle &= \int \Theta |\mathbf{p}\rangle \langle \mathbf{p}|\alpha\rangle^* \, d^3p \\ &= \int |-\mathbf{p}\rangle \langle \mathbf{p}|\alpha\rangle \, d^3p\end{aligned}$$

Theres nothing to keep us from moving the minus sign:

$$\begin{aligned}\Theta |\alpha\rangle &= \int |\mathbf{p}\rangle \langle -\mathbf{p}|\alpha\rangle^* \, d^3p \\ &= \int \phi^*(-\mathbf{p}) |\mathbf{p}\rangle \, d^3p,\end{aligned}$$

So we see that

$$\Theta\phi(\mathbf{p}) = \phi^*(-\mathbf{p}).$$

Does this make sense? Yes. Because in momentum-space time reversal means motion reversal; its intuitive that $\mathbf{p}$ picks up the negative sign, and we know from the class notes that the time reversal operator involves taking the complex conjugate, as we see in our answer.