Question

A particle of mass m and charge e_0 feels a weak constant electric field $\vec{E} = E_0\hat{x}$. Additionally, the particle experiences a constant strong magnetic field $\vec{B} = B_0\hat{z}$. The magnetic field strength is greater than the electric field strength, $E_0 < B_0$.

Describe a reference frame where either E or B is zero. State the velocity (both magnitude and direction) of that frame, and describe the fields in that frame.

Solve the solution for the Schrödinger equation in the Lorentz shifted reference frame for t=0.

What would be the properties of the solution in the original reference frame?

1 Relativistic Shift

We start off with an electric field in the x direction $\mathbf{E}_x$ and a magnetic field in the z direction $\mathbf{B}_z$. The equation for field transformations are

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel}$$
$$\mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \vec{v} \times \mathbf{B})$$
$$\mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel}$$
$$\mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - \frac{1}{c^2}\vec{v} \times \mathbf{E})$$

The electric field is weaker than the magnetic field, so we should use a relativistic shift that makes $E = 0$. This can be accomplished quite simply through a lorentz boost in the $\hat{y}$ direction.

The field transformation equations for a boost in the $\hat{y}$ direction with a magnetic field in the $\hat{z}$ direction are

$$\mathbf{E}'_x = \gamma(\mathbf{E}_x + v_0\mathbf{B}_z)$$
$$\mathbf{E}'_y = \mathbf{E}_y$$
$$\mathbf{E}'_z = \gamma\mathbf{E}_z$$

All the direction of E are zero, except in the x direction. If we set that to zero,

$$\mathbf{E}'_x = 0 = \gamma(\mathbf{E}_x + v_0\mathbf{B}_z)$$
$$\Rightarrow 0 = \mathbf{E}_x + v_0\mathbf{B}_z$$
$$v_0 = \frac{-\mathbf{E}_x}{\mathbf{B}_z}.$$

Hence, we can think of this as a particle in a frame moving with velocity $\frac{-\mathbf{E}_x}{\mathbf{B}_z}$ plus a circular motion inside the classical framework.

(Question: What would happen if E » B?)

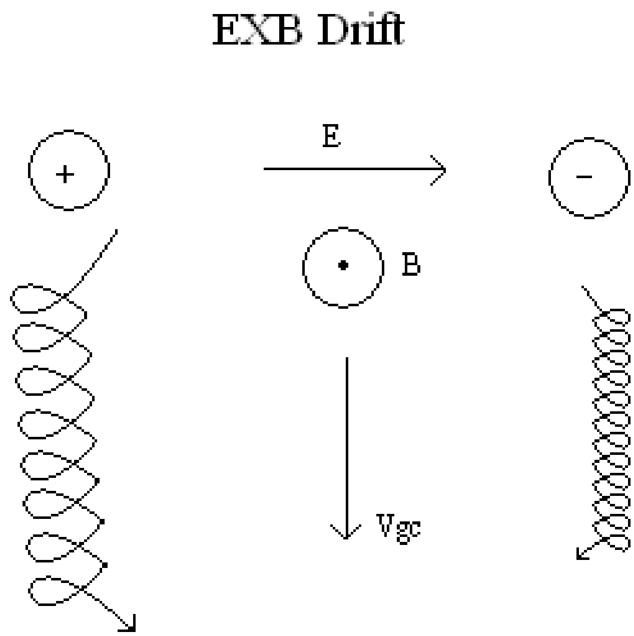

Figure 1.1: Classical motion

This lorentz boost changes the magnetic field strength as follows...

$$
\begin{aligned}
\mathrm{B}'_z &= \gamma(\mathrm{B}_z + \frac{1}{c^2} v_y E_x) \\
&= \gamma(\mathrm{B}_z - \frac{1}{c^2} \frac{E_x}{B_z} E_x) \\
&= \gamma \mathrm{B}_z \left(1 - \left(\frac{E_x}{B_z c}\right)^2\right) \\
&= \gamma \mathrm{B}_z \left(1 - \beta^2\right) \\
\mathrm{B}'_z &= \frac{\mathrm{B}_z}{\gamma}
\end{aligned}
$$

The magnetic field is weakened by a factor of γ.

2 Defining the Electromagnetic Four-Potential

(Note, we are now in the lorentz shifted space. I should put primes on all spatial variables, velocities, and functions dependent on them, but that is annoying to write and read, so just assume they are there.)

We just need to find Φ and $\vec{A}$. $\mathrm{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial A}{\partial t}$ so Φ will be zero because $E = 0$ and there is no time dependence. Defining $\vec{A}$ is trickier, but in this case, not too bad. It just needs to satisfy $\vec{B} = \nabla \times \vec{A}$, which, even though I hate to say it, I will do by observation. $\mathrm{B}_z = \frac{\partial}{\partial x}\mathrm{A}_y - \frac{\partial}{\partial y}\mathrm{A}_x$ so

$$\Phi = 0$$
$$\vec{A} = -B_z'\, y\hat{x}$$

(This is not the only correct answer for $\vec{A}$, I just wanted $\vec{A}$ to have only an x component.)

3 Solving the (Time Independent) Schrödinger Equation

First, let's define a wavefunction. We can simply use a separation of variables. (Trust me, it will end up working.)

$$\psi = \phi_x(x)\phi_y(y)\phi_z(z)$$

Our Schrödinger equation will end up being the following...

$$\begin{aligned}
\mathrm{H}\psi &= \frac{1}{2m}\left(-i\hbar\nabla - \frac{e_0}{c}\vec{A}\right)^2\psi \\
&= \frac{1}{2m}\left(-\hbar^2(\nabla\cdot\nabla\psi) + \frac{ie_0\hbar}{c}(\nabla\cdot\vec{A}\psi) + \frac{ie_0\hbar}{c}(\vec{A}\cdot\nabla\psi) + \frac{e_0^2}{c^2}(\vec{A}\cdot\vec{A}\psi)\right) \\
&= \frac{1}{2m}\left(-\hbar^2\nabla^2 + \frac{ie_0\hbar}{c}(\nabla\cdot\vec{A}) + 2\frac{ie_0\hbar}{c}(\vec{A}\cdot\nabla) + \frac{e_0^2}{c^2}(\vec{A}\cdot\vec{A})\right)\psi \\
&= \frac{1}{2m}\left(-\hbar^2\partial_x^2 - \hbar^2\partial_y^2 - \hbar^2\partial_z^2 - 2\frac{ie_0\hbar}{c}B_z' y\partial_x + \frac{e_0^2}{c^2}{B_z'}^2 y^2\right)\psi
\end{aligned}$$

Now, one can check and see that both $[\hat{P}_x, H] = 0$ and $[\hat{P}_z, H] = 0$, which means we can set $\phi_x = e^{ik_x x}$ and $\phi_z = e^{ik_z z}$, solutions of a free particle. See if you can figure out why? (Hint, solve $[\hat{P}_x, \hat{T} + \hat{V}] = 0$)

Now, we can plug in those wave functions and cancel out ϕ_x and ϕ_z and move on...

$$\mathrm{E}\phi_y = \frac{-\hbar^2}{2m}\partial_y^2\phi_y + \frac{h^2k_z^2}{2m}\phi_y + \frac{1}{2m}(\hbar^2k_x^2 + \frac{2\hbar k_x e B_0' y}{c} + \frac{e^2 B_0' y^2}{c^2})\phi_y$$
$$\mathrm{E}\phi_y = \frac{-\hbar^2}{2m}\partial_y^2\phi_y + \frac{h^2k_z^2}{2m}\phi_y + \frac{e_0^2{B_0'}^2}{2mc^2}\left(\frac{\hbar^2k_x^2c^2}{e_0^2{B_0'}^2} + \frac{2\hbar k_x c y}{e_0 B_0'} + y^2\right)\phi_y$$

If you do some annoying unit analysis, you see that you can add a convenient new argument...

$$y_0 = \frac{-\hbar k_x c}{e_0 B_0'}$$

$$E\phi_y = \frac{\hbar^2 k_z^2}{2m}\phi_y - \frac{\hbar^2}{2m}\partial_y^2\phi_y + \frac{e_0^2 {B_0'}^2}{2mc^2}\left(y' - y_0'\right)^2\phi_y$$

The first term is a free particle moving in the z direction, and the second term a harmonic oscillator in the y dimension. But where is the x direction here? Well, you can see by analyzing the eigenstate of the $\hat{P}_x$ operator.

$$\hat{P}_x\psi = (-i\hbar\partial_x - \frac{e_0}{c}A_x)\psi$$
$$mv_x = \hbar k_x + \frac{e_0}{c}B_0' y'$$

This results in a circular motion if one solves for the equation of motion. (See lecture notes equation 3.17) In quantum mechanics however, this better corresponds to a 2D harmonic oscillator. BUT, our wavefunction's form only has a 1D harmonic oscillator component. Quantum mechanics sure is funky. So, our solvable wave function becomes a plane wave in the z direction multiplied by a plane wave in the x direction multiplied by a 1D harmonic oscillator in the in the y direction, centered around a y coordinate that depends on the inital x momentum, with $\omega = (e_0 B_0')/(mc)$. (An interesting system, as the harmonic oscillator is represented purely by bound states and the free particle exist in a continuum, and the energy is QUANTIZED in the same way as a 1D harmonic oscillator, even though classically we would expect the system to be oscillating in a way involving 2 dimensions.

$$\psi' = e^{ik_{z'}x'}e^{ik_{z'}z'}\phi_n(y')$$
$$E = \frac{\hbar^2 k_z^2}{2m} + \hbar\omega(n + \frac{1}{2})$$
$$\omega = \frac{e_0 B_0'}{mc}$$

4 Return to Reference Frame

In the y direction, we will have length contraction because we boosted in the y direction, making the harmonic oscillator skinnier. We would also have modify the center of the harmonic oscillator to move at the velocity we used to Lorentz shift. (Unfortunately, it turns out there is no simple way to bring a wavefunction back through a Lorentz shift, be we can at least take estimates on what the new density would look like.)