S Scattering Presentation

Jeremiah Rowland and Mack Smith

1 Introduction

Low energy scattering hinges on the partial wave expansion. Since slow moving particles $\frac{\hbar^2 k^2}{2m} \ll V$ must have a large impact parameter b in order to have a large angular momentum,

$$L = \hbar k b_{max} \tag{1}$$

the incoming particle must have a small angular momentum in order to collide with the target. Thus, it suffices to consider only the s wave $l = 0$ in the scattering behavior. For a fast moving particle, $\frac{\hbar^2 k^2}{2m} \gg V$ see the Born approximation.

2 A classic problem: Scattering off of an attractive well

In the context of the final exam, you can expect to see something along the lines of the classic simple examples for s wave scattering, as the problems are often very simple or very lengthy. So to begin, we will discuss scattering off of an attractive potential. The aim of the calculation is to find the phase shift $\delta(k)$, the scattering length a_0, and the cross section σ

$$V(r) = \begin{cases} -V_0 & 0 < r < R \\ 0 & R < r \end{cases} \tag{2}$$

The radial equation for u is

$$\left(\frac{d^2}{dr^2} + 1\right)u_l(r) = \frac{l(l+1)}{r^2}u_l(x) + \beta u_l(x) \tag{3}$$

Or for s wave scattering, $l = 0$ gives

$$\left(\frac{d^2}{dr^2} + 1\right)u_0(r) = \beta u_0(r) \tag{4}$$

where $\beta = \frac{2m}{\hbar^2 k^2} V(k/r)$ The solutions look like the semi-infinite square well with a phase shift, which is obtained by requiring $u(0) = 0$,

$$u(r) = \begin{cases} A\sin(\kappa r) & 0 < r < R \\ B\sin(kr + \delta) & R < r \end{cases} \tag{5}$$

where $k = \frac{\sqrt{2mE}}{\hbar}$ and $\kappa = \frac{\sqrt{2m(E-V)}}{\hbar}$ The C^1 continuous condition gives

$$\begin{cases} A\sin(\kappa R) = B\sin(kR+\delta) \\ A\kappa\cos(\kappa R) = Bk\cos(kR+\delta) \end{cases} \tag{6}$$

After finding the phase shift, you could use this to find the relationship between A and B. If not asked for these parameters, you could skip this previous step.

In order to find the phase shift, we have to ensure the logarithmic derivative is continuous. The logarithmic derivative is defined as

$$\frac{\frac{d}{dr} r u_l(r)}{r u_l(r)} \tag{7}$$

Employing this condition, we obtain:

$$\frac{A\sin(\kappa R) + A\kappa R\cos(\kappa R)}{ARsin(\kappa R)} = \frac{B\sin(kR+\delta) + BRk\cos(kR+\delta)}{BR\sin(kR+\delta)}$$
$$\frac{1}{R} + \kappa\cot(\kappa R) = \frac{1}{R} + k\cot(kR+\delta)$$
$$\kappa\cot(\kappa R) = k\cot(kR+\delta)$$
$$k\tan(\kappa R) = \kappa\tan(kR+\delta)$$

And we come to the final result

$$\delta(k) = \arctan\left(\frac{k}{\kappa}\tan(kR)\right) - kR \tag{8}$$

The next step is to determine the scattering length which is defined as:

$$\lim_{k\to 0^+} \frac{d}{dk}\delta(k) \tag{9}$$

Evaluating the derivative, we obtain

$$\frac{d}{dk}\delta = \frac{\frac{\arctan(kR)}{\kappa} + \frac{kR}{\kappa(R^2k^2+1)}}{\frac{k^2\arctan^2(Rk)}{\kappa^2} + 1} - R \tag{10}$$

and immediately we obtain the result for the scattering length:

$$a = -\lim_{k\to 0^+} \frac{d}{dk}\delta(k) = R \tag{11}$$

The limit is apparent as numerator goes to 0 and the denominator goes to 1. Employing the relation for the cross section

$$\sigma = 4\pi a^2 = 4\pi R^2 \tag{12}$$

which means we have recovered hard sphere scattering the low energy case, which makes sense as when $k \to 0$, $\frac{V}{k} \to -\infty$ thus effectively returning the hard (attractive) sphere case.

3 Problem 2: Scattering off of a soft spherical shell

For the soft spherical shell, we will find the same parameters: $\delta(k), a_0,$ and σ. The potential is given by

$$V(r) = -\beta\delta(r-a) \tag{13}$$

The wave function is similar to the previous problem,

$$u(r) = \begin{cases} A\sin(kr) & 0 < r < a \\ \sin(kr+\delta) & a < r \end{cases} \tag{14}$$

To begin, we note that the logarithmic derivative continuity condition is not helpful, as the delta function does not satisfy this boundary condition. In particular evaluating the logarithmic derivative condition gives,

$$\begin{aligned} \frac{A\sin(ka) + AR\cos(ka)}{Aasin(ka)} &= \frac{\sin(ka+\delta) + AR\cos(ka+\delta)}{Aasin(ka+\delta)} \\ \frac{1}{a} + k\cot(ka) &= \frac{1}{a} + k\cot(ka+\delta) \\ ka &= ka+\delta \\ \delta &= 0 \end{aligned}$$

Which does not incorporate the influence of the delta function, so the phase shift is 0.

Instead, we should use the boundary conditions for the delta function potential, which is that it must be C^0 continuous and the discontinuity in the derivative must satisfy

$$-\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\psi(x)\big|_{a+\epsilon} - \frac{\partial}{\partial x}\psi(x)\big|_{a-\epsilon}\right) = \beta\psi(a) \tag{15}$$

These conditions give

$$\begin{cases} A\sin(ka) = \sin(ka+\delta) \\ kA\cos(ka) - \frac{2m\beta}{\hbar^2}A\sin(ka) = k\cos(ka+\delta) \end{cases} \tag{16}$$

Note that we have chosen $\psi(a) = \psi(a-\epsilon)$ to eliminate a second delta. Further, we have rearranged the terms such that delta is isolated on the RHS. Dividing (2) by (1) and rearranging, and introducing $c = \frac{-2m\beta}{\hbar^2}$, we obtain the first result

$$\delta(k) = \operatorname{arccot}\left(\cot(ka) - \frac{c}{k}\right) - ka \tag{17}$$

Taking the derivative, we obtain

$$\frac{d}{dk}\delta(k) = \frac{a\csc^2(ka) + \frac{c}{k^2}}{\left(\cot(ka) + \frac{c}{k}\right)^2 + 1} \tag{18}$$

Upon evaluating the limit, we find

$$\frac{d}{dk}\delta(k)\big|_{k=0} = \frac{a}{ac+1} - a \tag{19}$$

which gives

$$a_0 = -\frac{d}{dk}\delta(k)\big|_{k=0} = a - \frac{a}{ac+1} = \frac{a^2c}{ac+1} \tag{20}$$

and the scattering cross section is then given by

$$\sigma = 4\pi a_0^2 = 4\pi\left(\frac{a^2c}{ac+1}\right)^2 \tag{21}$$

which notably in the case $c \to 0$ returns the trivial case calculated above