PHY851: Diffraction and Form Factors

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Subject Exam Spring 2020 : Problem 3

Consider a beam of particles of momentum $\hbar k$ elastically scattering off three identical targets placed at the following positions:

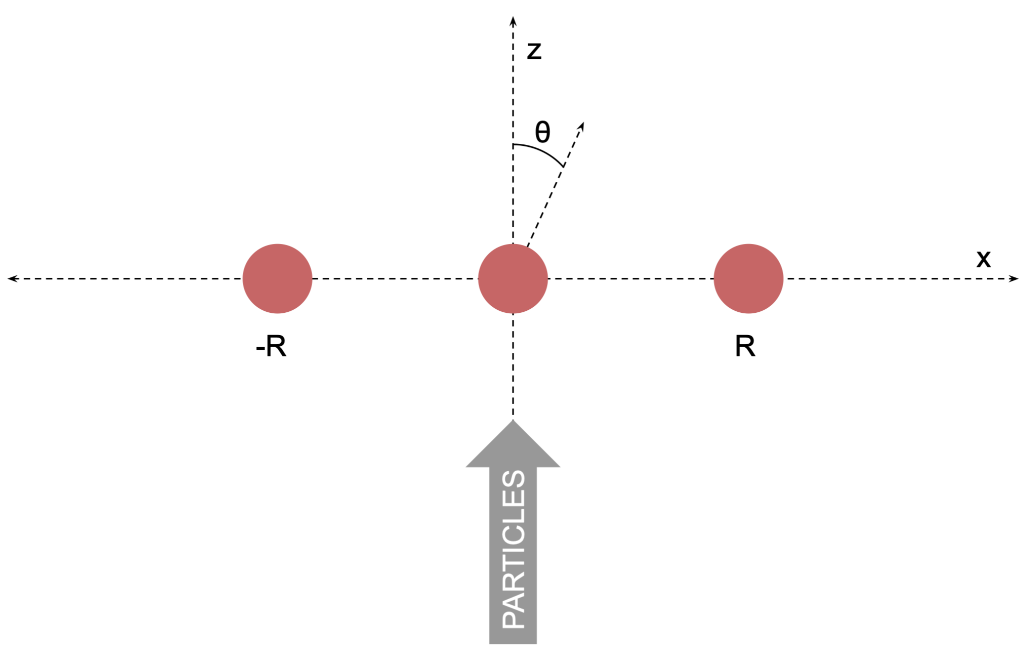
$$\begin{aligned} \vec{R}_1 &= (x = 0, y = 0, z = 0) \\ \vec{R}_2 &= (x = R, y = 0, z = 0) \\ \vec{R}_3 &= (x = -R, y = 0, z = 0) \end{aligned} \tag{1}$$

The direction of the scattered particles is denotes in spherical coordinates, with θ describing the direction relative to the beam (z) axis, and ϕ measuring the direction relative to the x axis in the $x - y$ plane, i.e. if the wave number for the scattered particle is $\vec{k}^{(f)}$, then

$$k_z^{(f)} = k^{(f)} \cos\theta, \qquad k_x^{(f)} = k^{(f)} \sin\theta\cos\phi, \qquad k_y^{(f)} \sin\theta\sin\phi \tag{2}$$

1. Consider scattering observed in the $x - z$ plane ($\phi = 0$). At what polar angles θ will the differential cross section disappear?
2. Repeat for scattering observed in the $y - z$ plane ($\phi = 90°$).

The situation as described is shown in the following figure.



Consider scattering observed in the $x - z$ plane ($\phi = 0$). At what polar angles θ will the differential cross section disappear?

Step 1 : Determine expression for differential cross section
Because each scattering center is identical, we know that the differential cross section can be expressed as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{m^2}{4\pi^2\hbar^4}|v(\vec{k})|^2\tilde{S}(\vec{k}) \tag{3}$$

(equation 7.15 in lecture notes) where $\tilde{S}(\vec{k})$ is the Fourier transform of the structure function. For our own convenience, we can define

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_{\text{isolated}} = \alpha(\vec{k}) \tag{4}$$

So that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \alpha(\vec{k})\tilde{S}(\vec{k}) \tag{5}$$

Step 2 : Determine expression for structure function
By definition,

$$\tilde{S}(\vec{k}) = \sum_{\delta\vec{a}} e^{i\vec{k}\cdot\delta\vec{a}} \tag{6}$$

We know that our structure has three scattering centers, located at $(-R, 0, 0)$, $(0, 0, 0)$, and $(R, 0, 0)$. Therefore

$$\begin{aligned}\tilde{S}(\vec{k}) &= 1 + e^{-iRk_x} + e^{iRk_x} \\ &= 1 + 2\cos Rk_x \\ &= 1 + 2\cos\left(Rk\sin(\theta)\cos(\phi)\right)\end{aligned} \tag{7}$$

As given, we are looking for scattering observed in the $x - z$ plane, which forces $\phi = 0$. This implies that

$$\tilde{S}(\vec{k}) = 1 + 2\cos\left(Rk\sin(\theta)\right) \tag{8}$$

Step 3 : Solve for zeros of the differential cross section
We can safely assume that the potential from each scattering center is nonzero at any point that we are considering. This implies that $\alpha(\vec{k}) \neq 0$ for all $\vec{k}$. Therefore

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= 0 \\ \implies \alpha(\vec{k})\tilde{S}(\vec{k}) &= 0 \\ \implies \tilde{S}(\vec{k}) &= 0\end{aligned} \tag{9}$$

Plugging in our expression from step 2,

$$\begin{aligned}1 + 2\cos(Rk\sin(\theta)) &= 0 \\ \cos(Rk\sin(\theta)) &= \frac{-1}{2} \\ Rk\sin(\theta) &= \cos^{-1}\left(\frac{-1}{2}\right) \\ \theta &= \sin^{-1}\left(\frac{1}{Rk}\cos^{-1}\left(\frac{-1}{2}\right)\right)\end{aligned} \tag{10}$$

From here, we have two potential solutions,

$$\theta = \sin^{-1}\left(\frac{1}{Rk}\left(\frac{2\pi}{3} + 2\pi n\right)\right) \qquad \theta = \sin^{-1}\left(\frac{1}{Rk}\left(\frac{4\pi}{3} + 2\pi m\right)\right) \tag{11}$$

For $n, m \in \mathbb{Z}$.
It is important to note here that not all n, m are valid in this solution. Mathematically, $\sin^{-1}(x)$ is only defined for $-1 \leq x \leq 1$, so n will have an upper and lower bound based on R and k.
Additionally, from a physical perspective, θ is defined to be the angle from the beam path. Therefore, θ only takes on values such that $0 \leq \theta \leq \pi$. This implies that

$$\begin{aligned} 0 \leq \frac{1}{Rk}\left(\frac{2\pi}{3} + 2\pi n\right) \leq 1 \implies 0 \leq n + \frac{1}{3} \leq \frac{Rk}{2\pi} \\ 0 \leq \frac{1}{Rk}\left(\frac{4\pi}{3} + 2\pi m\right) \leq 1 \implies 0 \leq m + \frac{2}{3} \leq \frac{Rk}{2\pi} \end{aligned} \tag{12}$$

Which concludes the problem.

Repeat for scattering observed in the $y - z$ plane ($\phi = 90°$).

In the $y - z$ plane, $\phi = \pi/2$. Using this and the equation found previously for the Fourier transform of the structure function:

$$\tilde{S}(\vec{k}) = 1 + 2\cos\left(Rk\sin(\theta)\cos(\phi)\right) \tag{13}$$

we want to find θ where the differential cross section is zero. Given that $\cos(\pi/2) = 0$, we get the following equation for the structure function:

$$\begin{aligned} \tilde{S}(\vec{k}) &= 1 + 2\cos\left(Rk\sin(\theta)\cos(\phi)\right) \\ 0 &= 1 + 2\cos\left(Rk\sin(\theta)\cos\left(\frac{\pi}{2}\right)\right) \\ 0 &= 1 + 2\cos\left(0\right) \end{aligned} \tag{14}$$

Thus it can be seen that there are no angles for which the differential cross section disappears in the $y - z$ plane.