Fermi's Golden Rule

Jared Dechant, Samuel Sottile

Mary and Bob in an infinite square well

A *bob* particle ψ_b is in the ground state of a one dimensional infinite potential well of length a,

$$V_0 = \begin{cases} 0 & \text{if } 0 < x < a \\ \infty & \text{else} \end{cases}.$$

A perturbation is added that allows a *bob* particle to undergo a transformation into a *mary* particle ψ_m. The *mary* particle does not feel the effects of the square well potential. The *bob* and *mary* particles have the same mass m. The perturbation is of the form,

$$V_{bm} = \langle \psi_m | V_{bm} | \psi_b \rangle = g \int dx \psi_b^*(x) \partial_x \psi_m(x),$$

where g is the interaction strength between the two particles.

Calculate the rate of transition between a *bob* and *mary* particle.

Solution

The ground state wave-function ψ_b of an asymmetric 1D infinite potential well and the free particle wave-function ψ_m are of the form:

$$\begin{aligned} \psi_b(x) &= \sqrt{\frac{2}{a}} \sin \frac{\pi}{a} x \\ \psi_m(x) &= \frac{1}{\sqrt{L}} e^{ikx} \end{aligned}$$

with corresponding energies E_b and E_m

$$\begin{aligned} E_b &= \frac{\hbar^2 \pi^2}{2ma^2} \\ E_m &= \frac{\hbar^2 k^2}{2m} \end{aligned}$$

Thus the interaction V_{bm} can be calculated as

$$\begin{aligned} V_{bm} &= g \int dx \psi_b^*(x) \partial_x \psi_m(x) \\ &= \sqrt{\frac{2}{La}} g(ik) \int_0^a dx \sin \frac{\pi}{a} x e^{ikx} \\ &= \sqrt{\frac{2}{La}} g(ik) \int_0^a dx \frac{1}{2i} \left(e^{i(\pi+ka)x/a} - e^{i(\pi-ka)x/a} \right) \\ &= \sqrt{\frac{2}{La}} g(ik) \frac{\pi a (1 + e^{iak})}{\pi^2 - a^2 k^2} \end{aligned}$$

Now we will calculate the transition rate using Fermi's Golden Rule. Recall

$$R_{i\to n} = \frac{2\pi}{\hbar}|V_{ni}|^2\delta(\epsilon_n - \epsilon_i)$$

So

$$\begin{aligned} R_{b\to m} &= \frac{2\pi}{\hbar}|V_{bm}|^2\delta(\frac{\hbar^2k^2}{2m} - \frac{\hbar^2\pi^2}{2ma^2}) \\ &= \frac{2\pi}{\hbar}\frac{\pi^2a^2}{(\pi^2-a^2k^2)^2}(2+2\cos ak)\frac{2}{La}g^2k^2\frac{2m}{\hbar^2}\delta(k^2-\frac{\pi^2}{a^2}) \\ &= \frac{16m\pi^3ag^2k^2(1+\cos ak)\delta(k^2-\frac{\pi^2}{a^2})}{\hbar^3L(\pi^2-a^2k^2)^2} \end{aligned}$$

where we used the identity

$$\delta(\alpha x) = \frac{1}{\alpha}\delta(x).$$

Also, note the other delta function identity:

$$\delta(x^2-\alpha^2) = \frac{1}{2\alpha}[\delta(x-\alpha)+\delta(x+\alpha)]$$

Integrating over all momentum states:

$$\begin{aligned} R &= \frac{16m\pi^3g^2a}{\hbar^3L}\int\frac{L}{2\pi}dk\frac{k^2}{(\pi^2-a^2k^2)^2}(1+\cos ak)\delta(k^2-\frac{\pi^2}{a^2}) \\ &= \frac{16m\pi^3g^2a}{\hbar^3L}\int\frac{L}{2\pi}dk\frac{k^2}{(\pi^2-a^2k^2)^2}(1+\cos ak)\frac{a}{2\pi}(\delta(k-\frac{\pi}{a})+\delta(k+\frac{\pi}{a})) \\ &= \frac{8m\pi^2g^2a}{\hbar^3}(2)\frac{a}{2\pi}\frac{\pi^2/a^2}{8\pi^2} \end{aligned}$$

The factor of 2 comes from the fact that the delta function is non zero when $k = \pm\frac{\pi^2}{a^2}$ (everything else only depends on $|k|$). Also note that the $\frac{1}{8\pi^2}$ comes from the limit

$$\lim_{t\to\pi}\frac{1+\cos(t)}{(\pi^2-t^2)^2} = \frac{1}{8\pi^2}$$

So, simplifying, we get

$$R = \frac{m\pi g^2}{\hbar^3}$$

The units work out as g is in units of Jm and $\hbar$ has units of Js. Interestingly, the rate is independent of the width of the potential well.