Adding Angular Momentum

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Problem: Express the state $|S = \frac{1}{2}, L = 1, m_s = \frac{1}{2}, m_l = 0 >$ as a linear combination of eigenstates of total angular momentum J and projection m_j.

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Strategy:

1. Find the state with largest J and m_j and write it in terms of S, L, m_s, and m_l
2. Apply the lowering operator $J_- = S_- + L_-$ the needed number of times to create new states of J and m_j that are linear combinations of S, L, m_s, and m_l
3. Use the orthogonality of $|J, m_j>$ states to find new states of J and m_j that are linear combinations of S, L, m_s, and m_l
4. Using the states from steps 2 and 3, find a linear combination of them that results in $|S = \frac{1}{2}, L = 1, m_s = \frac{1}{2}, m_l = 0>$ expressed in terms of $|J, m_j>$ states

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We use the simple rules for adding two angular momenta to find the possible results for this system to be

$$J = \frac{1}{2}, \frac{3}{2}, \qquad m_j = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}.$$

Our state is then

$$|J = \frac{3}{2}, m_j = \frac{3}{2} >= |S = \frac{1}{2}, L = 1, m_s = \frac{1}{2}, m_l = 1 > .$$

Written more succinctly:

$$|J = \frac{3}{2}, m_j = \frac{3}{2} >= |m_s = \frac{1}{2}, m_l = 1 > .$$

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The lowering operator for J is

$$J_-|J, m_j >= \hbar\sqrt{J(J+1) - m_j(m_j - 1)}|J, m_j - 1 >$$

and those for S and L are analogous. Applying $J_- = S_- + L_-$ to the state we previously found:

$$J_-|J = \frac{3}{2}, m_j = \frac{3}{2} >= (S_- + L_-)|m_s = \frac{1}{2}, m_l = 1 >$$

$$|J = \frac{3}{2}, m_j = \frac{1}{2} >= \sqrt{\frac{1}{3}}|m_s = \frac{-1}{2}, m_l = 1 > +\sqrt{\frac{2}{3}}|m_s = \frac{1}{2}, m_l = 0 >$$

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By orthogonality of $|J, m_j >$ states, we know that

$$|J = \frac{3}{2}, m_j = \frac{1}{2} >= \sqrt{\frac{1}{3}}|m_s = \frac{-1}{2}, m_l = 1 > +\sqrt{\frac{2}{3}}|m_s = \frac{1}{2}, m_l = 0 >$$

is orthogonal to

$$|J = \frac{1}{2}, m_j = \frac{1}{2} >= \alpha|m_s = \frac{-1}{2}, m_l = 1 > +\beta|m_s = \frac{1}{2}, m_l = 0 > .$$

Solving

$$< J = \frac{3}{2}, m_j = \frac{1}{2}|J = \frac{1}{2}, m_j = \frac{1}{2} >= 0$$

we find

$$\alpha = -\beta\sqrt{2}.$$

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The $|J, m_j >$ states are normalized. Thus:

$$< J = \frac{1}{2}, m_j = \frac{1}{2} | J = \frac{1}{2}, m_j = \frac{1}{2} >= \alpha^2 + \beta^2 = 1$$

$$\Longrightarrow$$

$$2\beta^2 + \beta^2 = 1$$

$$\beta = \sqrt{\frac{1}{3}}$$

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We finally have the following two states

$$|J = \frac{3}{2}, m_j = \frac{1}{2} >= \sqrt{\frac{1}{3}}|m_s = \frac{-1}{2}, m_l = 1 > + \sqrt{\frac{2}{3}}|m_s = \frac{1}{2}, m_l = 0 >,$$

$$|J = \frac{1}{2}, m_j = \frac{1}{2} >= -\sqrt{\frac{2}{3}}|m_s = \frac{-1}{2}, m_l = 1 > + \sqrt{\frac{1}{3}}|m_s = \frac{1}{2}, m_l = 0 > .$$

Multiply the top state by the $\sqrt{2}$ and add it to the bottom state to find our answer

$$|m_s = \frac{1}{2}, m_l = 0 >= \sqrt{\frac{2}{3}}|J = \frac{3}{2}, m_j = \frac{1}{2} > + \sqrt{\frac{1}{3}}|J = \frac{1}{2}, m_j = \frac{1}{2} > .$$

Problem: An electron is in an $l = 1$ state of a hydrogen atom. It experience a spin-orbit interaction

$$V_{s.o.} = \alpha \vec{L} \cdot \vec{S}$$

and feels an external magnetic field

$$V_b = \mu \vec{B} \cdot (\vec{L} + 2\vec{S})$$

Find the energy eigenvalues.

Step 1: $V_{s.o.}$ in the $|j, m_j\rangle$ basis

Using $\vec{J} = \vec{L} + \vec{S}$,

$$V_{s.o.} = \frac{\alpha}{2}(|\vec{J}|^2 - |\vec{L}|^2 - |\vec{S}|^2)$$
$$= \frac{\alpha\hbar^2}{2}(j(j+1) - \frac{11}{4})$$
$$= \frac{\alpha\hbar^2}{2}\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & -2 & \\ & & & & & -2 \end{pmatrix}$$

where $|\frac{3}{2}, \frac{3}{2}\rangle = \begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}$, $|\frac{3}{2}, \frac{1}{2}\rangle = \begin{pmatrix}0\\1\\0\\0\\0\\0\end{pmatrix}$, $|\frac{3}{2}, \frac{-1}{2}\rangle = \begin{pmatrix}0\\0\\1\\0\\0\\0\end{pmatrix}$, $|\frac{3}{2}, \frac{-3}{2}\rangle = \begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix}$, $|\frac{1}{2}, \frac{1}{2}\rangle = \begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix}$, $|\frac{1}{2}, \frac{-1}{2}\rangle = \begin{pmatrix}0\\0\\0\\0\\0\\1\end{pmatrix}$

Step 2: V_b in the $|m_l, m_s\rangle$ basis

$$V_b = \mu\hbar B \begin{pmatrix} 2 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & 0 & \\ & & & & & -2 \end{pmatrix}$$

where $|1, \frac{1}{2}\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $|1, \frac{-1}{2}\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $|0, \frac{1}{2}\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $|0, \frac{-1}{2}\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $|-1, \frac{1}{2}\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $|-1, \frac{-1}{2}\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Step 3: Finding the Clebsch-Gordan Matrix

$$|j, m_j\rangle = \langle m_l, m_s | j, m_j\rangle \, |m_l, m_s\rangle = C \, |m_l, m_s\rangle$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\frac{1}{3}} & -\sqrt{\frac{2}{3}} & 0 \end{pmatrix}$$

Note that $C^T C = 1$

Step 4: Transforming V_b from $|m_l, m_s\rangle$ to the $|j, m_j\rangle$ basis

$$(V_b)_{j,m_j} = CV_bC^T$$

$$= \mu\hbar B \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 & \frac{-\sqrt{2}}{3} & 0 \\ 0 & 0 & \frac{-2}{3} & 0 & 0 & \frac{-\sqrt{2}}{3} \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & \frac{-\sqrt{2}}{3} & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{-\sqrt{2}}{3} & 0 & 0 & \frac{-1}{3} \end{pmatrix}$$

Step 5: Full Hamiltonian in the $|j, m_j\rangle$ basis

$$H = V_{s.o.} + (V_b)_{j,m_j}$$

$$= \begin{pmatrix} \frac{\alpha\hbar^2}{2} + 2\mu\hbar B & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha\hbar^2}{2} + \frac{2\mu\hbar B}{3} & 0 & 0 & \frac{-\sqrt{2}\mu\hbar B}{3} & 0 \\ 0 & 0 & \frac{\alpha\hbar^2}{2} - \frac{2\mu\hbar B}{3} & 0 & 0 & \frac{-\sqrt{2}\mu\hbar B}{3} \\ 0 & 0 & 0 & \frac{\alpha\hbar^2}{2} - 2\mu\hbar B & 0 & 0 \\ 0 & \frac{-\sqrt{2}\mu\hbar B}{3} & 0 & 0 & -\alpha\hbar^2 + \frac{\mu\hbar B}{3} & 0 \\ 0 & 0 & \frac{-\sqrt{2}\mu\hbar B}{3} & 0 & 0 & -\alpha\hbar^2 - \frac{\mu\hbar B}{3} \end{pmatrix}$$

Eigenvalues

$$\epsilon_1 = \frac{\alpha\hbar^2}{2} + 2\mu\hbar B$$

$$\epsilon_2 = \frac{\alpha\hbar^2}{2} - 2\mu\hbar B$$

$$\epsilon_3 = \frac{-\alpha\hbar^2}{4} + \frac{\mu\hbar B}{2} + \sqrt{(\frac{\alpha\hbar^2}{4} + \frac{\mu\hbar B}{2})^2 + \frac{\alpha^2\hbar^4}{2}}$$

$$\epsilon_4 = \frac{-\alpha\hbar^2}{4} + \frac{\mu\hbar B}{2} - \sqrt{(\frac{\alpha\hbar^2}{4} + \frac{\mu\hbar B}{2})^2 + \frac{\alpha^2\hbar^4}{2}}$$

$$\epsilon_5 = \frac{-\alpha\hbar^2}{4} - \frac{\mu\hbar B}{2} + \sqrt{(\frac{\alpha\hbar^2}{4} - \frac{\mu\hbar B}{2})^2 + \frac{\alpha^2\hbar^4}{2}}$$

$$\epsilon_6 = \frac{-\alpha\hbar^2}{4} - \frac{\mu\hbar B}{2} - \sqrt{(\frac{\alpha\hbar^2}{4} - \frac{\mu\hbar B}{2})^2 + \frac{\alpha^2\hbar^4}{2}}$$

Eigenvalues Plot