Chapter 9 – Homework Solutions

1. As part of an elaborate calculation using Fermi's Golden rule, you find yourself needing to calculate the following matrix element squared

$$|\mathcal{M}_{fi}|^2 = |\langle f|\Psi_A^\dagger(\vec{r})\Psi_B(\vec{r})|i\rangle|^2.$$

The initial state i is composed of N_B particles of type B, which are all in the same single-particle state of momentum $\vec{k}_B$. The final state $|f\rangle$ is composed of $N_B - 1$ particles of type B, in the same level $\vec{k}_B$, along with one particle of type A with momentum $\vec{k}_A$. The momentum states are defined within some large volume V.

(a) Find $|\mathcal{M}_{fi}|^2$. The momentum states are defined within some large volume V.

(b) Repeat but in this case assume the N_B particles are all in different momentum states, $\vec{k}_n,\ n = 1 \cdots N_B$, with the same values in the final state, with the exception of $\vec{k}_1$, which is missing.

Solution:
a)

$$\begin{aligned}
\mathcal{M} &= \langle 0|\frac{a_A(\vec{k}_A)(a_B(\vec{k}_B))^{N_B-1}}{\sqrt{(N_B-1)!}}\Psi_A^\dagger(\vec{r})\Psi_B(\vec{r})\frac{(a_B^\dagger)^{N_B}}{\sqrt{N_B!}}|0\rangle \\
&= \frac{e^{i\vec{k}_B\cdot\vec{r}-i\vec{k}_A\cdot\vec{r}}}{V\sqrt{N_B!(N_B-1)!}}\langle 0|a_A(\vec{k}_A)a_A^\dagger(\vec{k}_A)(a_B(\vec{k})^{N_B})(a_B^\dagger(\vec{k}))^{N_B}|0\rangle \\
&= \frac{N_B!e^{i\vec{k}_B\cdot\vec{r}-i\vec{k}_A\cdot\vec{r}}}{V\sqrt{N_B!(N_B-1)!}} \\
&= \frac{\sqrt{N_B}}{V}e^{i\vec{k}_B\cdot\vec{r}-i\vec{k}_A\cdot\vec{r}}, \\
|\mathcal{M}|^2 &= \frac{N_B}{V^2}.
\end{aligned}$$

b)

$$\mathcal{M} = \langle 0|\left[\prod_{m=2\cdots N_B} a_B(\vec{k}_m)a_A(\vec{k}_A)\right]\Psi_A^\dagger(\vec{r})\Psi_B(\vec{r})\left[\prod_{m=1\cdots N_B} a_B^\dagger(\vec{k}_m)\right]|0\rangle$$

One can see that after expanding $\Psi_B(\vec{r})$ in momentum states that only the $\vec{k}_1$ term contributes, and that when expanding $\Psi_A(\vec{r})$ that only the $\vec{k}_A$ term contributes. Thus,

$$\begin{aligned}
\mathcal{M} &= \frac{e^{i\vec{k}_B\cdot\vec{r}-i\vec{k}_A\cdot\vec{r}}}{V}\langle 0|\left[\prod_{m=2\cdots N_B} a_B(\vec{k}_m)a_A(\vec{k}_A)\right] a_A^\dagger(\vec{k}_A)a_B(\vec{k}_1)\left[\prod_{m=1\cdots N_B} a_B^\dagger(\vec{k}_m)\right]|0\rangle \\
&= \frac{e^{i\vec{k}_B\cdot\vec{r}-i\vec{k}_A\cdot\vec{r}}}{\sqrt{V}}, \\
|\mathcal{M}|^2 &= \frac{1}{V^2}.
\end{aligned}$$

If one summed over all the $|\mathcal{M}|^2$ for all final states with different choices of which momenta $\vec{k}_i$ were missing, then the answer would have been N/V.

2. Consider b-particles of mass m confined by a one-dimensional harmonic oscillator potential characterized by a frequency ω. The b particles interact with massless and spinless a-particles through their respective field operators,

$$H_{\text{int}} = g \int dx \Psi^\dagger(x)\Phi(x)\Psi(x),$$

where Φ and Ψ are the field operators for the a-particles and b-particles respectively. Assume the b particles are sufficiently heavy to ignore their recoil energy.

$$\begin{aligned}\Phi(x) &= \frac{1}{\sqrt{L}} \sum_k \frac{1}{\sqrt{E_k}} \left(e^{-ikx} a_k^\dagger + e^{ikx} a_k\right)\\ \Psi^\dagger(x) &= \frac{1}{\sqrt{L}} \sum_k e^{-ikx} b_k^\dagger,\end{aligned}$$

(a) What is the dimension of g?

(b) What is the decay rate of a b particle in the first excited state.

Solution:

a) Units of Φ are $1/\sqrt{LE}$, units of Ψ are $1/\sqrt{L}$.

$$\begin{aligned}[E] &= [g][L]\frac{1}{[L^{3/2}][E^{1/2}]},\\ [g] &= [E]^{3/2}[L]^{1/2}.\end{aligned}$$

b)

$$\begin{aligned}\Gamma &= \frac{2\pi}{\hbar} \left|\langle i|V|f\rangle\right|^2 \delta(\hbar\omega - \hbar kc),\\ \langle n=0,k|V|n=1\rangle &= g \int dx\ \psi_1^*(x)\langle k|\Phi(x)|0\rangle\psi_0(x)\\ &= g \int dx\ \psi_1^*(x)\psi_0(x)\frac{1}{\sqrt{L}}\frac{1}{\sqrt{E_k}}e^{ikx}\\ e^{ikx} &= e^{ik(b+b^\dagger)\sqrt{\hbar/2m\omega}} = e^{ik\sqrt{\hbar/2m\omega}b^\dagger}e^{ik\sqrt{\hbar/2m\omega}b}e^{-k^2\hbar/4m\omega}.\end{aligned}$$

Now, use the fact that $e^{iAb}|0\rangle = |0\rangle$, which you can see by expanding the exponential. Then expand the other exponential but only keep one power of $b^\dagger$ as all other powers give zero. Then,

$$\begin{aligned}
\langle 1|e^{ikx}|0\rangle &= ik\sqrt{\frac{\hbar}{2m\omega}}e^{-\hbar k^2/4m\omega}, \\
\langle n=0,k|V|n=1\rangle &= \frac{g}{\sqrt{LE_k}}ik\sqrt{\frac{\hbar}{2m\omega}}e^{-\hbar k^2/4m\omega}, \\
\Gamma_k &= \frac{g^2}{LE_k}\frac{\hbar k^2}{2m\omega}e^{-\hbar k^2/2m\omega}\frac{2\pi}{\hbar}\delta(\hbar\omega-\hbar kc), \\
\Gamma &= L\int_0^\infty \frac{dk}{\pi}\frac{g^2\pi k^2}{LE_k m\omega}e^{-\hbar k^2/2m\omega}\delta(\hbar ck-\hbar\omega) \\
&= \frac{g^2k^2}{m\hbar ck\hbar c\omega}e^{-\hbar k^2/2m\omega} \\
&= \frac{g^2}{m\hbar^2c^3}e^{-\hbar k^2/2m\omega}.
\end{aligned}$$

Checking units, (use the fact that $[\hbar]=[E][t]$ and $[mc^2]=[E]$)

$$[\Gamma]=[g]^2\frac{1}{[E^3][T^2][L]/[T]}=\frac{E^3L}{[E^3][T^2][L]/[T]}=\frac{1}{[T]} \quad \checkmark$$

3. Show that Eq. (**??**) is satisfied by using the electric and magnetic fields defined in Eq. (**??**). Note: After squaring $\vec{E}$ and $\vec{B}$, ignore any cross terms when you involving rapid oscillations in time, i.e. those that behave as $e^{\pm 2iE_k t/\hbar}$

   From notes, the 2 equations are

$$\int d^3r \frac{\vec{E}^2 + \vec{B}^2}{8\pi} = \sum_{\vec{k},s} E_k \left(a^\dagger_{\vec{k},s} a_{\vec{k},s} + \frac{1}{2} \right), \tag{0.1}$$

   and

$$\vec{A}(\vec{r},t) = \sqrt{\frac{2\pi\hbar^2 c^2}{V}} \sum_{k,s} \frac{1}{\sqrt{E_k}} \left(\vec{\epsilon}_s(\vec{k}) e^{i\vec{k}\cdot\vec{r} - iE_k t/\hbar} a_{k,s} + \vec{\epsilon}^{\,*}_s(\vec{k}) e^{-i\vec{k}\cdot\vec{r} + iE_k t/\hbar} a^\dagger_{k,s} \right). \tag{0.2}$$

Solution:
Let

$$\frac{1}{c}\frac{\partial \vec{A}}{\partial t} = \vec{E} = \sqrt{\frac{2\pi}{V}} \sum_k E_k^{1/2} (\vec{\epsilon}_s(\vec{k}) a_{k,s} e^{i\vec{k}\cdot\vec{r} - iE_k t/\hbar} + \vec{\epsilon}^{\,*}_s(\vec{k}) a^\dagger_{k,s} e^{-i\vec{k}\cdot\vec{r} + iE_k t\hbar}),$$

$$\vec{B} = \nabla \times \vec{A} = -i\sqrt{\frac{2\pi}{V}} \sum_k E_k^{1/2} ((\vec{\epsilon}_s(\vec{k}) \times \hat{k}) a_{k,s} e^{i\vec{k}\cdot\vec{r} - iE_k t/\hbar} - (\vec{\epsilon}^{\,*}_s(\vec{k}) \times \hat{k}) a^\dagger_{k,s} e^{-i\vec{k}\cdot\vec{r} + iE_k t/\hbar}),$$

The energy density is

$$\int d^3r \, \frac{|\vec{E}|^2 + |\vec{B}|^2}{8\pi} = 2\pi \sum_{kk'} \frac{1}{V} \int d^3r \; e^{i(\vec{k}-\vec{k}')\cdot\vec{r}} (E_k E_{k'})^{1/2} \left[\vec{\epsilon}^{\,*}_s \cdot \vec{\epsilon}_{s'} + (\hat{k} \cdot \vec{\epsilon}^{\,*}_s)(\hat{k}' \cdot \vec{\epsilon}'_s) \right] (a^\dagger_k a_{k'} + a_k a^\dagger_{k'}).$$

Note that terms that behave as $e^{\pm 2iEt/\hbar}$ have been left out.
Use the facts that

$$\frac{1}{V} \int d^3r \; e^{i(\vec{k}-\vec{k}')\cdot\vec{r}} = \delta_{kk'},$$

and

$$\vec{\epsilon}_s \cdot \vec{\epsilon}_{s'} = \delta_{ss'},$$
$$(\hat{k} \times \vec{\epsilon}_s) \cdot (\hat{k} \times \vec{\epsilon}_{s'}) = \delta_{ss'},$$

to obtain

$$\int d^3r \; \frac{|\vec{E}|^2 + |\vec{B}|^2}{8\pi} = \sum_k \frac{2\pi}{8\pi} 2E_k (a^\dagger_k a_k + a_k a^\dagger_k)$$
$$= \sum_k E_k (a^\dagger_k a_k + \frac{1}{2}).$$

4. A proton in a nucleus decays from an excited state to its ground state by emitting a photon of momentum $\hbar\vec{k}$ and polarization $\vec{\epsilon}_s$. The matrix element describing the decay is

$$\langle 0, k, s|V|1\rangle = \beta\vec{\epsilon}_s \cdot \int d^3r \frac{e^{-i\vec{k}\cdot\vec{r}}}{\sqrt{V}} \left(\phi_0^*(\vec{r})\nabla\phi_1(\vec{r}) - [\nabla\phi_0^*(\vec{r})]\phi_1(\vec{r})\right).$$

The factor β absorbed all the various factors involved in defining the vector field in Eq. (**??**). Assume the ground and excited states are modeled with a three-dimensional harmonic oscillator of frequency ω. If the excited state is in the first level of a harmonic oscillator and has an angular momentum projection m, what is the shape ($\theta\phi$ dependence) of the angular distribution of the photons, $d\Gamma/d\Omega$, for each m. Assume that the wavelength of the photon is sufficiently long that the phase $e^{i\vec{k}\cdot\vec{r}} \approx 1$. Remember that the two polarizations of the photon must be perpendicular to $\vec{k}$. You need only calculate the angular shape of the distribution – ignore the prefactors.

Some help: The first excited state of the harmonic oscillator is three-fold degenerate. In the Cartesian basis these have the form $\sim x\phi_0, y\phi_0$ and $z\phi_0$, where ϕ_0 is the ground-state wave function. These can be mapped to three states that are eigenstates of angular momentum, $\ell = 1; m = 1, 0, -1$ as discussed in Chapter **??**. The wave functions of states with $m = \pm 1$ have to have an angular dependence given by $Y_{1\pm 1} \sim \sin\theta e^{\pm i\phi}$, whereas the wavefunction for $m = 0$ has to be proportional to $Y_{10} \sim \cos\theta$. Using the fact that $r\cos\theta = z$ and $r\sin\theta e^{\pm i\phi} = x \pm iy$, the $m = \pm 1$ wave functions are proportional to $x \pm iy$, whereas the $m = 0$ wave is proportional to z.

Solution:

$$\begin{aligned}
i\hbar\partial_z &= \sqrt{\frac{\hbar m\omega}{2}} i(a_z - a_z^\dagger), \\
\langle n=0|\partial_z|n_z=1\rangle &= \sqrt{\frac{m\omega}{2\hbar}}, \\
|n=1, m=0\rangle &= |n_x=0, n_y=0, n_z=1\rangle, \\
|n=1, m=\pm 1\rangle &= \frac{1}{\sqrt{2}}|n_x=1, n_y=n_z=0\rangle \pm i\frac{1}{\sqrt{2}}|n_x=n_z=0, n_y=1\rangle, \\
\langle n=0, m=0|\partial_i|n=1, m=0\rangle &= \sqrt{\frac{m\omega}{2\hbar}}\delta_{iz}, \\
\langle n=0, m=0|\partial_i|n=1, m=\pm 1\rangle &= \sqrt{\frac{m\omega}{2\hbar}}\left(\frac{1}{\sqrt{2}}\delta_{ix} + i\frac{1}{\sqrt{2}}\delta_{iy}\right) \\
\vec{\epsilon}\cdot\langle n=0|\vec{\nabla}|n=1, m=0\rangle &= \sqrt{\frac{m\omega}{2\hbar}}\epsilon_z, \\
\vec{\epsilon}\cdot\langle n=0|\vec{\nabla}|n=1, m=\pm 1\rangle &= \sqrt{\frac{m\omega}{2\hbar}}\frac{1}{\sqrt{2}}(\epsilon_x + i\epsilon_y)
\end{aligned}$$

The $n = 1, m = 0$ decays go as

$$\sum_s (\vec{\epsilon}_s(\vec{k}) \cdot \hat{z})^2 = 1 - (\hat{k} \cdot \hat{z})^2 = \sin^2 \theta,$$

The $n = 1, m = \pm 1$ decays go as

$$\begin{aligned} \frac{1}{2} \sum_s |\vec{\epsilon}_s \cdot (\hat{x} \pm i\hat{y})|^2 &= 1 - \frac{|\hat{k} \cdot (\hat{x} \pm i\hat{y})|^2}{2} \\ &= 1 - \frac{1}{2} \left| \frac{k_x \pm i k_y}{k} \right|^2 \\ &= 1 - \frac{1}{2} \sin^2 \theta = \frac{1}{2} + \frac{\cos^2 \theta}{2}. \end{aligned}$$

You may note that if you sum over all 3 polarizations, the angular dependence is uniform.

5. A spinless particle of mass M and charge e is in the first excited state of a three-dimensional harmonic oscillator characterized by a frequency ω. Assume the particle is in the Cartesian state of a harmonic oscillator with $n_z = 1$, i.e. $m = 0$. Using the interaction

$$H_{\text{int}} = \vec{j} \cdot \vec{A}/c,$$

(a) Calculate the decay rate of the charged particle into the ground state of the oscillator in the dipole approximation.

(b) Calculate $d\Gamma/d\Omega$ as a function of the emission angles of the photon, θ and ϕ.

(c) In terms of the unit vectors $\hat{k}$, $\hat{\theta}$ and $\hat{\phi}$, the two polarization vectors which are allowed for emission of a photon at an angle θ, ϕ are $\hat{\theta}$ and $\hat{\phi}$. For each of these two polarization vectors, calculate $d\Gamma_s/d\Omega$, the probability of decaying via emission of a photon emitted in the θ, ϕ direction with polarization s.

Solution:
From lecture notes,

$$\Gamma = \frac{4e^2 k}{3\hbar m^2 c^2} |\mathcal{M}|^2.$$

In dipole approximation,

$$\begin{aligned}
\mathcal{M}_z &= im\omega \int d^3r \ \psi_f(\vec{r}) z \psi_i(\vec{r}), \\
\psi_f &= |0\rangle, \quad \psi_i = a_z^\dagger |0\rangle, \\
z &= \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger), \\
\mathcal{M}_z &= im\omega \sqrt{\frac{\hbar}{2m\omega}} = i\sqrt{\hbar m\omega/2}, \\
\Gamma &= \frac{4e^2 k}{3\hbar m^2 c^2} \hbar m\omega/2 \\
&= \frac{2e^2\omega^2}{3mc^3}.
\end{aligned}$$

b) From lecture notes,

$$\begin{aligned}
\frac{d\Gamma}{d\Omega} &= \frac{e^2 k}{2\pi\hbar m^2 c^2} \left\{ |\mathcal{M}|^2 - |\hat{k} \cdot \mathcal{M}|^2 \right\} \\
&= \frac{e^2 k}{2\pi\hbar m^2 c^2} \frac{\hbar m\omega}{2} (1 - (\hat{k} \cdot \hat{z})^2) \\
&= \frac{e^2\omega^2}{4\pi mc^3} \sin^2\theta.
\end{aligned}$$

c)

$$
\begin{aligned}
\vec{\epsilon} &= (\vec{\mathcal{M}} - \hat{k}(\hat{k} \cdot \vec{\mathcal{M}}))/|\vec{\mathcal{M}}| \\
\vec{\mathcal{M}} &= |\vec{\mathcal{M}}|\hat{z}, \\
\frac{d\Gamma_s}{d\Omega_k} &= \frac{e^2 k}{2\pi\hbar m^2 c^2}|\vec{\mathcal{M}}|^2(\hat{M} \cdot \hat{\epsilon}_s)^2 \\
&= \frac{e^2 k}{2\phi\hbar m^2 c^2}|\vec{\mathcal{M}}|^2(\hat{z} \cdot \hat{\epsilon}_s)^2 \\
\hat{z} &= \hat{r}\cos\theta - \hat{\theta}\sin\theta, \\
\frac{d\Gamma_{\hat{\theta}}}{d\Omega_k} &= \frac{e^2 k|\vec{\mathcal{M}}|^2}{2\pi\hbar m^2 c^2}\sin^2\theta = \frac{e^2\omega^2}{4\pi m c^3}\sin^2\theta, \\
\frac{d\Gamma_{\hat{\phi}}}{d\Omega_k} &= 0.
\end{aligned}
$$

6. Again consider a spinless particle of mass M and charge e n the first excited state of a three-dimensional harmonic oscillator characterized by a frequency ω. However, this time assume the charged particle is originally in a state with angular momentum projection $m = +1$ along the z axis. Using the interaction

$$H_{\rm int} = \vec{j} \cdot \vec{A}/c,$$

and applying the dipole approximation,

(a) Find the decay rate Γ of the first excited state.
(b) Find the differential decay rate $d\Gamma/d\Omega$.
(c) Describe the polarization of a photon emitted in the $\hat{x}$ direction.
(d) Describe the polarization vector of a particle emitted in the $\hat{z}$ direction.

Solution:
a) Like previous problem but replace $\hat{z}$ with $(\hat{x} + i\hat{y})/\sqrt{2}$.

$$\vec{\mathcal{M}} = -i\frac{\sqrt{\hbar m\omega}}{2}(\hat{x} + i\hat{y})$$

So, $|\vec{\mathcal{M}}|^2$ is the same and

$$\Gamma = \frac{2e^2\omega^2}{3mc^3}.$$

b)

$$\begin{aligned}
\frac{d\Gamma}{d\Omega_k} &= \frac{e^2 k}{2\pi\hbar m^2 c^2}\left\{|\vec{\mathcal{M}}|^2 - |\hat{k}\cdot\vec{\mathcal{M}}|^2\right\} \\
&= \frac{e^2\omega^2}{4\pi mc^3}\left\{1 - \frac{1}{2}|\hat{k}\cdot(\hat{x} + i\hat{y})|^2\right\} \\
&= \frac{e^2\omega^2}{4\imath mc^3}\left\{1 - \frac{1}{2}\sin^2\theta\right\}.
\end{aligned}$$

c)

$$\begin{aligned}
\vec{\epsilon} &\sim \frac{\vec{\mathcal{M}}}{|\vec{\mathcal{M}}} - \frac{\hat{k}(\hat{\mathcal{M}}\cdot\hat{k})}{|\vec{\mathcal{M}}|} \\
&\sim -\frac{\hat{x} + i\hat{y}}{\sqrt{2}} + \frac{\hat{k}(\hat{k}\cdot(\hat{x} + i\hat{y}))}{\sqrt{2}}.
\end{aligned}$$

If $\hat{k} = \hat{x}$,

$$\begin{aligned}
\vec{\epsilon} &\sim -\frac{(\hat{x} + i\hat{y})}{\sqrt{2}} + \frac{\hat{x}}{\sqrt{2}} = i\frac{\hat{y}}{\sqrt{2}}, \\
\vec{\epsilon} &= \hat{y} \quad \text{within a phase.}
\end{aligned}$$

d)

$$\begin{aligned}\vec{\epsilon} &\sim -\frac{(\hat{x}+i\hat{y})}{\sqrt{2}} - \hat{z}(\hat{z}\cdot(\hat{x}+i\hat{y}) \\ &= -\frac{\hat{x}+i\hat{y}}{\sqrt{2}}, \\ \vec{\epsilon} &= \frac{\hat{x}+i\hat{y}}{\sqrt{2}} \quad \text{within a phase.}\end{aligned}$$