# Chapter 7 – Homework Solutions

1. Using the Born approximation estimate the differential scattering cross section, $d\sigma/d\Omega$ for particles of mass $m$ scattering off the following potentials.

   (a) $\mathcal{V}(\vec{r}) = V_0\Theta(a-r)$.
   (b) $\mathcal{V}(\vec{r}) = a^3 V_0 \delta^3(\vec{r})$.
   (c) $\mathcal{V}(\vec{r}) = a^3 V_0 \left[\delta^3(\vec{r}-a\hat{z}) + \delta^3(\vec{r}+a\hat{z})\right]$.
   (d) $\mathcal{V}(\vec{r}) = a^3 V_0 \left[\delta^3(\vec{r}-a\hat{z}) - \delta^3(\vec{r}+a\hat{z})\right]$.
   (e) $\mathcal{V}(\vec{r}) = a^3 V_0 \left[\delta^3(\vec{r}-a\hat{x}) - \delta^3(\vec{r}+a\hat{x})\right]$.
   (f) $V_0 e^{-r/a}/r$.

   You can express your answer either in terms of the momentum transfer $\vec{q}$, or in terms of the beam momentum $k$ and the scattering angle $\theta, \phi$.

**Solution**:

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{m^2}{4\pi^2\hbar^4}\left|\int d^3r\ \mathcal{V}(r)e^{i(\vec{k}_f-\vec{k}_i)\cdot\vec{r}}\right|^2\\ &= \frac{m^2}{4\pi^2\hbar^4}\left|\tilde{\mathcal{V}}(\vec{q})\right|^2\end{aligned}$$

We will confine ourselves to finding expressions for $\mathcal{V}(\vec{q})$. a)

$$\begin{aligned}\tilde{\mathcal{V}}(\vec{q}) &= \int_{r<a} 2\pi r^2 dr d\cos\theta\ V_0 e^{iqr\cos\theta}\\ &= 2\pi V_0\int_0^a r^2 dr\frac{e^{iqr}-e^{-iqr}}{iqr}\\ &= \frac{4\pi V_0}{q}\int_0^a rdr\ \sin(qr)\\ &= \frac{4\pi V_0}{q}\left\{\int_0^a \frac{dr}{q}\cos(qr)\quad - \frac{a}{q}\cos(qq)\right\}\\ &= 4\pi V_0\left\{\frac{\sin(qa)}{q^3} - \frac{a}{q^2}\cos(qa)\right\}.\end{aligned}$$

b)

$$V_0 a^3.$$

c)

$$\begin{aligned}\tilde{\mathcal{V}}(\vec{q}) &= V_0 a^3\left\{e^{iq_z a} + e^{-iq_z a}\right\}\\ &= 2V_0 a^3\cos(ka(1-\cos\theta)).\end{aligned}$$

d)

$$\tilde{\mathcal{V}}(\vec{q}) = 2iV_0 a^3\sin(ka(1-\cos\theta)).$$

e)

$$\tilde{\mathcal{V}}(\vec{q}) = -2iV_0V_0\sin(ka\sin\theta\cos\phi),$$

f)

$$\begin{aligned}
\tilde{\mathcal{V}}(\vec{q}) &= 2\pi V_0\int_0^\infty rdr\ e^{-r/a}\int_{-1}^1 d\cos\theta\ e^{iqr\cos\theta} \\
&= \frac{4\pi V_0}{q}\int_0^\infty dr\ \ \sin(qr)e^{-r/a} \\
&= \frac{2\pi V_0}{iq}\int dr\left(e^{-r/a+iqr} - e^{-r/a-iqr}\right) \\
&= \frac{2\pi V_0}{iq}\left(\frac{1}{(1/a)-iq} - \frac{1}{(1/a)+iq}\right) \\
&= \frac{4\pi V_0}{q^2+(1/a)^2}.
\end{aligned}$$

2. By taking two derivatives of the form factor at $q = 0$,

$$\frac{\partial}{\partial q_i}\frac{\partial}{\partial q_j}F(\vec{q})\bigg|_{q=0},$$

one can generate moments of the charge distribution,

$$\langle (r_i - \bar{r}_i)(r_j - \bar{r}_j)\rangle \equiv \int d^3r\ \rho(\vec{r})(r_i - \bar{r}_i)(r_j - \bar{r}_j).$$

(a) Working in a coordinate system where $\bar{r}_i = 0$, prove the relation above.

(b) Test your answer by comparing to the result of Example **??**. First calculate $\langle r_i r_j\rangle$ by integrating to get a weighted average of $r_i r_j$ using $\rho_q(\vec{r})$, then compare to the expression above using derivatives of the form factor.

**Solution**:
a)

$$\begin{aligned}
F(\vec{q}) &= \frac{1}{Ze}\int d^3r\ \rho(\vec{r})e^{i\vec{q}\cdot\vec{r}},\\
\frac{\partial}{\partial q_i}\frac{\partial}{\partial q_j}F(\vec{q})\bigg|_{q=0} &= \frac{1}{Ze}\int d^3r\ \rho(\vec{r}(-r_i r_j)e^{i\vec{q}\cdot\vec{r}}\\
&= \frac{-1}{Ze}\langle r_i r_j\rangle\int d^3r\rho(\vec{r})\\
&= -\langle r_i r_j\rangle.
\end{aligned}$$

b) For a Gaussian where

$$\rho(\vec{r}) = Ze\frac{1}{(2\pi R^2)^{3/2}}e^{-r^2/2a^2},$$

The form factor is

$$F(\vec{q}) = e^{-q^2a^2/2},$$

and

$$\begin{aligned}
\frac{\partial}{\partial q_i}\frac{\partial}{\partial q_j}F(\vec{q})\bigg|_{q=0} &= \frac{\partial}{\partial q_i}q_j a^2 e^{-q^2a^2/2}\bigg|_{q=0}\\
&= a^2\delta_{ij} \quad \checkmark
\end{aligned}$$

3. The cross section for scattering a particle with momentum $\hbar k$ off a single target is

$$\frac{d\sigma}{d\Omega} = \alpha,$$

which is independent of the scattering angle. Here, we assume that the cross section is small. Now, two targets are placed a distance $a$ apart, separated along the $z$ axis (the same axis along which the incident beam is directed). At what scattering angles does the differential cross section, $d\sigma/d\Omega$, equal zero? (This is worked out as an example in the notes)

**Solution**:
The cross section will be

$$\alpha \left| \sum_a e^{i\vec{q}\cdot\vec{r}_a} \right|^2,$$

where $\vec{r}_a$ is the position of each of the scatterers and $\vec{q}$ is the momentum transfer,

$$q_z = k(1-\cos\theta), \quad q_x = -k\sin\theta\cos\phi, \quad q_y = -k\sin\theta\sin\phi.$$

The differential cross section will be proportional to

$$\frac{d\sigma}{d\Omega} \propto \left|1 + e^{iq_z a}\right|^2.$$

So, there will be zeros when

$$\begin{aligned} q_z a &= (2n+1)\pi, \\ k(1-\cos\theta_n)a &= (2n+1)\pi, \\ \cos\theta_n &= 1 - \frac{1}{ka}(2n+1)\pi. \end{aligned}$$

4. The cross section for scattering a particle with momentum $\hbar k$ off a single target is

$$\frac{d\sigma}{d\Omega} = \alpha,$$

which is independent of the scattering angle. Here, we assume that the cross section is small. Now, two targets are placed a distance $a$ apart, separated along the $x$ axis (perpendicular to the axis along which the incident beam is directed). At what scattering angles does the differential cross section, $d\sigma/d\Omega$, equal zero? Specify both the polar angle $\theta$ and the azimuthal angle $\phi$.

**Solution**:
The cross section will be

$$\alpha \left| \sum_a e^{i\vec{q}\cdot\vec{r}_a} \right|^2,$$

where $\vec{r}_a$ is the position of each of the scatterers and $\vec{q}$ is the momentum transfer,

$$q_z = k(1-\cos\theta), \quad q_x = -k\sin\theta\cos\phi, \quad q_y = -k\sin\theta\sin\phi.$$

The differential cross section will be proportional to

$$\frac{d\sigma}{d\Omega} \propto \left|1 + e^{iq_x a}\right|^2.$$

So, there will be zeros when

$$\begin{aligned} q_x a &= (2n+1)\pi, \\ ka\sin\theta_n\cos\phi_n &= (2n+1)\pi, \\ \sin\theta_n\cos\phi_n &= \frac{1}{ka}(2n+1)\pi. \end{aligned}$$

Thus, lines in the $\theta - \phi$ plane of zero scattering are mapped out. Note that in the $xz$ plane that there are zeros, but that in the $yz$ plane there are no zeros because $\cos\phi$ would be zero in that plane.

5. Consider a charge $Z$ that is uniformly distributed within a sphere of radius $R$. A

   (a) Find the squared form factor $F(\vec{q})$.

   (b) For an incoming wave with wave number $k\hat{z}$ what is $F$ as a function of the scattering angle $\theta$? Express $F$ in terms of $kR$ and $\theta$.

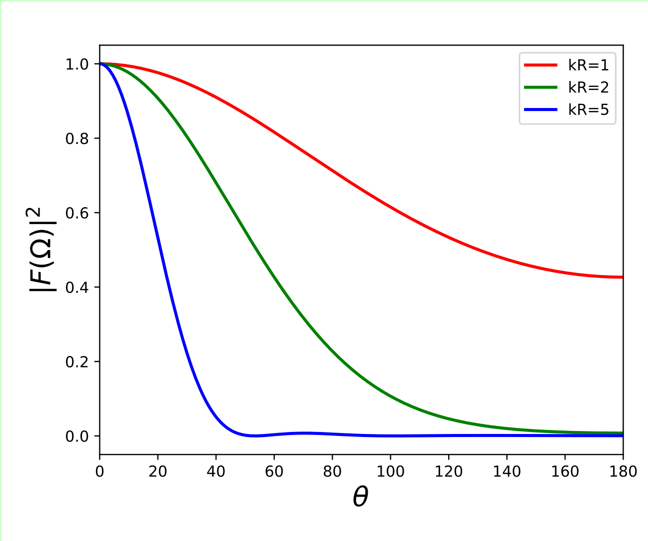
   (c) Plot $|F|^2$ from (b) for the cases where $kR = 1, 2$ and $5$ as a function of $\theta$.

**Solution**:
a)

$$\begin{aligned}
F &= \frac{1}{Z}\int d^3r\ e^{i\vec{q}\cdot\vec{r}}\rho(\vec{r}) \\
&= \frac{2\pi\rho}{Z}\int_0^R r^2dr\ d\cos\theta\ e^{iqr\cos\theta} \\
&= \frac{4\pi\rho}{Z}\int_0^R drdr\ \frac{\sin(qr)}{q} \\
&= \frac{4\pi\rho}{Zq^2}\left\{\int_0^R dr\ \ \cos(qr) - r\cos(qr)|_0^R\right\} \\
&= \frac{4\pi\rho}{Zq^2}\left(\frac{\sin(qR)}{q} - R\cos(qR)\right) \\
&= \frac{3}{(qR)^3}\left(\sin(qR) - qR\cos(qR)\right).
\end{aligned}$$

Here we used $Z = 4\pi\rho R^3/3$.
b)

$$\begin{aligned}
&q_z = k(1-\cos\theta),\quad q_x = k\sin\theta\cos\phi,\quad q_y = k\sin\theta\sin\phi, \\
&q^2 = q_x^2 + q_y^2 + q_z^2, \\
&\ q = k\sqrt{2(1-\cos\theta)} \\
&\quad = 2k\sin(\theta/2), \\
&F = \frac{3}{(2kR\sin(\theta/2))^3}\left\{\sin(2kR\sin(\theta/2)) - 2kR\sin(\theta/2)\cos(2kR\sin(\theta/2))\right\}.
\end{aligned}$$




c)

6. A $\pi^+$, which is a spin-zero meson, scatters off a proton through a $\Delta^{++}$ resonance(which is comprised of three up quarks). The $\Delta^{++}$ is spin 3/2 baryon. The masses of the pion, proton and delta are 139.58 MeV/c$^2$, 938.28 MeV/c$^2$ and 1232 MeV/c$^2$ respectively. The width of the $\Delta$ is 120 MeV.

   (a) Using relativistic dispersion relations, $E = \sqrt{p^2c^2 + m^2}$, what is the relative momentum, $q$, of the pion and proton at resonance? This is one half the difference of the two momenta in the center-of-mass frame. I.e. $\epsilon_\pi(p_R) + \epsilon_p(p_R) = M_\Delta$.

   (b) Estimate the cross section at resonance? Give your answer in millibars. One mb equals $10^{-24}$ cm and 10 mb= 1 fm$^2$.

**Solution**:
a) From Eq. (7.49), and setting $E = E_R$, one can see that the maximum cross section is

$$\sigma_{\rm max} = \frac{(2S_\Delta + 1)}{(2S_p + 1)} \frac{4\pi}{k_R^2},$$

and the resonant wave number is given by $\hbar k_R = p_R$ and

$$M_R^2 = \sqrt{p_R^2 + m_\pi^2} + \sqrt{p_R^2 + m_p^2}.$$

Subtracting the pion energy from both sides, then squaring,

$$\begin{aligned}
(M_R^2 + m_\pi^2 + p_R^2 + 2M_R\sqrt{p_R^2 + m_\pi^2} &= p_R^2 + m_p^2, \\
4M_R^2(p_R^2 + m_\pi^2) &= (m_p^2 - m_\pi^2 - M_R^2)^2, \\
p_R^2 &= \frac{1}{4M_R^2}\left(-4M_R^2m_\pi^2 + m_R^4 + m_p^4 + m_\pi^4\right. \\
&\qquad \left. -2m_p^2m_\pi^2 - 2m_p^2M_R^2 + 2m_\pi^2m_R^2\right) \\
&= \frac{1}{4M_R^2}\left(M_R^4 + m_p^4 + m_\pi^4 - 2m_p^2m_\pi^2 - 2M_R^2m_\pi^2 - 2M_R^2m_p^2\right).
\end{aligned}$$

Plugging in: $p_R = 226.0$ MeV/$c$.
b) $k_R = 226/\hbar c$, with $\hbar c = 197.327$ MeV/$c$. So $k_R = 1.145$ fm$^{-1}$ and

$$\sigma_{\rm max} = \frac{8\pi}{k_R^2} = 19.2 \text{ fm}^2 = 192 \text{ mb}.$$

7. Consider a particle of mass $m$ that could be confined to a spherical well,

$$\mathcal{V}(r) = \begin{cases} 0, & r < a \\ V_0, & a < r < 2a \\ 0, & r > 2a \end{cases}$$

(a) Use the WKB method to estimate the decay rate of a particle of mass $m$ escaping from a spherical trap defined by the potential. Assume the barrier is sufficiently high to approximate the energy of the trapped particle with an infinite well.

(b) Find an expression to estimate the cross section for a particle scattering off the potential well with an energy near the ground state energy described above. You can give your answer as a function of the incident energy, $E$, $m$, $V_0$, $a$, and the width $\Gamma$.

**Solution**:
a) Use the energy for the ground state of a square well

$$E_R = \frac{\hbar^2\pi^2}{2ma^2}$$

Probability of tunneling is

$$\begin{aligned} P &= \exp\left\{-2\int_a^{2a} \frac{dx\sqrt{2m(V-E_R)}}{\hbar}\right\} \\ &= \exp\left\{-\frac{2a}{\hbar}\sqrt{2m(V_0-R_R)}\right\}. \end{aligned}$$

The decay rate is the trial rate, $v/2a$, multiplied by the tunneling success probability $P$,

$$\begin{aligned} \Gamma &= \frac{v}{2a}P \\ &= \frac{1}{2a}\sqrt{\frac{2E_R}{m}}\exp\left\{-\frac{2a}{\hbar}\sqrt{2m(V_0-R_R)}\right\}. \end{aligned}$$

Here, the velocity was $\sqrt{2E_R/m}$.
b) Assume a resonant form for scattering,

$$\sigma \approx \frac{4\pi}{k_R^2}\frac{(\Gamma/2)^2}{(E-E_R)^2/\hbar^2+(\Gamma/2)^2},$$

Here, the resonant wave number is given by

$$\begin{aligned} \frac{\hbar^2 k_R^2}{2m} &= E_R, \\ k_R &= \frac{2mE_R}{\hbar}. \end{aligned}$$

8. Consider the function

$$\tilde{f}(\omega) = \frac{i}{\omega - E/\hbar + i\Gamma/2}.$$

Show that the Fourier transform is

$$\begin{aligned} f(t) &= \frac{1}{2\pi} \int d\omega \; e^{-i\omega t} \tilde{f}(\omega) \\ &= e^{-iEt/\hbar - \Gamma t/2} \Theta(t). \end{aligned}$$

HINT: Use contour integration.

**Solution**:

$$f(t) = \frac{1}{2\pi} \int d\omega \; e^{-i\omega t} \frac{i}{\omega - E/\hbar + i\Gamma/2}$$

There is a pole at $\omega_R = E/\hbar - i\Gamma/2$, so the contour closes in the lower plane (clockwise contour), so you get $-2\pi i$ times the residue, which yields

$$f(t) = e^{-i\omega_R t} = e^{-i\omega_R t} = e^{-iEt/\hbar - \Gamma t/2}.$$