Chapter 5 – Homework Solutions

1. Let $\mathcal{T}_{\vec{d}}$ denote the translation operator (displacement vector $\vec{d}$); $\mathcal{D}(\hat{n}, \phi)$, the rotation operator; and π the parity operator. Which, if any, of the following pairs commute? Why?

   (a) $\mathcal{T}_{\vec{d}}$ and $\mathcal{T}_{\vec{d}'}$ ($\vec{d}$ and $\vec{d}'$ are in different directions.)
   (b) $\mathcal{D}(\hat{n}, \phi)$ and $\mathcal{D}(\hat{n}', \phi')$ ($\hat{n}$ and $\hat{n}'$ are in different directions.)
   (c) $\mathcal{T}_{\vec{d}}$ and Π.
   (d) $\mathcal{D}(\hat{n}, \phi)$ and Π.

   Solution:
   a) YES! ∂_i and ∂_j commute
   b) NO! The generators L_i and L_j don't commute (unless $i = j$)
   c) NO! It makes a difference if you translate then reflect or reflect then translate
   d) YES! L_i and Π commute. I.e. Π does not affect $\vec{L}$.

2. Because of weak (neutral-current) interactions there is a parity-violating potential between the atomic electron and the nucleus as follows:

$$V = \lambda \left[\delta^3(\vec{r}) \mathbf{S} \cdot \mathbf{p} + \mathbf{S} \cdot \mathbf{p} \delta^3(\vec{r}) \right]$$

where $\mathbf{S}$ and $\mathbf{p}$ are the spin and momentum operators of the electron, and the nucleus is assumed to be situated at the origin. As a result, the ground state of an alkali atom, usually characterized by $|n, \ell, j, m\rangle$ actually contains tiny contributions from other eigenstates as follows

$$|n, \ell, j, m\rangle \rightarrow |n, \ell, j, m\rangle + \sum_{n', \ell', j', m'} C_{n', \ell', j', m'} |n', \ell', j', m'\rangle$$

On the basis of symmetry considerations *alone*, what can you say about (n', ℓ', j', m') which give rise to non-vanishing contributions?

Solution:
1.) Because of angular momentum conservation one can say the $j' = j$ and $m' = m$.
2.) Because the operator has odd parity:
If ℓ is even then ℓ' is odd and
if ℓ is odd then ℓ' is even.
3.) Not from symmetry, but because the matrix element has a factor $\delta(\vec{r})$ and radial wave functions behave as r^ℓ, one can state that either
$\ell = 0$ and $\ell' = 1$ or $\ell = 1$ and $\ell' = 0$.

3. Suppose a spinless particle is bound to a fixed center by a potential $V(\vec{r})$ so asymmetrical that no two energy levels are degenerate. Using time-reversal invariance prove

$$\langle \mathbf{L} \rangle = 0.$$

for any energy eigenstate. Use the fact each eigenwave function $\psi(\vec{r})$ must be an eigenstate of the time reversal operator with eigenvalue $e^{i\gamma}$, thus $\psi^*(\vec{r}) = e^{i\gamma}\psi(\vec{r})$. Also use the fact that $\langle \alpha | L_i | \alpha \rangle$ is real because L_i is a Hermitian operator. (This is known as **quenching** of orbital angular momentum.)

Solution:
$\langle i|\vec{L}|i\rangle$ is real because $\vec{L}$ is Hermitian. Further,

$$\langle i|\vec{L}|i\rangle = \int d^r \ \psi_i^*(\vec{r}\vec{L}\psi_i(\vec{r})$$

Because there is no degeneracy, the time-reversed operator must return the same state within a phase,

$$\Theta\psi = e^{i\gamma}\psi.$$

Further, the time reversed operator must be the complex conjugate,

$$\Theta\psi = \psi^*.$$

Thus,

$$\psi^* = e^{i\gamma}\psi,$$

which implies $\psi = e^{-i\gamma}\psi^*$. Now,

$$\begin{aligned}
\langle i|L_z|i\rangle &= \int d^3r \ \psi_i^*(\vec{r}) i\hbar\partial_\phi \psi_i(\vec{r}) \\
&= \int d^3r \ \psi_i(\vec{r})(-i\hbar\partial_\phi)\psi_i^*(\vec{r}) \ \ \text{(because it is real)} \\
&= \int d^3r \ e^{-i\gamma}\psi^*(\vec{r}) i\hbar\partial_\phi \psi_i(\vec{r}) e^{i\gamma} \\
&= \int d^3r \ \psi_i^*(\vec{r})(-i\hbar\partial_\phi)\psi_i(\vec{r}).
\end{aligned}$$

The matrix element equals minus itself, so it must be zero. This can work for any L_i by defining angle around the axis i.

4. Consider the time-reversal operator for spin-1/2 particles, $\Theta = \sigma_y K$, where K takes the complex conjugate of all quantities to its right. Show that Θ commutes with the rotation operator,

$$\mathcal{R}(\vec{\theta}) = \cos(\theta) + i\vec{\sigma}\cdot\hat{\theta}\sin(\theta).$$

Solution:

$$[\Theta, i\sigma_x] = \sigma_y K(i\sigma_x) - (i\sigma_x)\sigma_y K$$

because σ_x is real

$$= -i\sigma_y\sigma_x K - i\sigma_x\sigma_y K$$
$$= 0 \quad \checkmark$$

$$[\Theta, i\sigma_y] = \sigma_y K(i\sigma_y) - (i\sigma_y)\sigma_y K$$

because σ_y is imaginary

$$= i\sigma_y^2 K - i\sigma_y^2$$
$$= 0 \quad \checkmark$$

$$[\Theta, i\sigma_z] = \sigma_y K(i\sigma_z) - (i\sigma_z)\sigma_y K$$

because σ_z is real

$$= -i\sigma_y\sigma_z K - i\sigma_z\sigma_y K$$
$$= 0 \quad \checkmark$$

$$\mathcal{R} = \cos\theta\mathbb{I} + i\sigma_x\hat{\theta}_x\sin\theta + i\sigma_y\hat{\theta}_y\sin\theta + i\sigma_z\hat{\theta}_z\sin\theta.$$

Θ commutes with $\mathcal{R}$ because it commutes with $\mathbb{I}$, σ_x, σ_y and σ_z.

5. Consider a particle of mass M confined to a two-dimensional circle of radius R.

   (a) Write down the Schrödinger equation for the wave function $\psi(\phi)$, where the potential depends only on ϕ, and radial motion is ignored.

   (b) Assuming the potential is periodic,

$$V(\phi + 2\pi/N) = V(\phi),$$

   where N is an integer. Write the boundary condition relating $\psi(\phi)$ and $\psi(\phi + 2\pi/N)$, where the eigenvalue of the rotation operator, $\mathcal{R}(2\pi/N)$, is $e^{i\gamma}$. What values of γ are allowed?

   (c) Assume the potential ,

$$V(\phi) = \beta \sum_{j=1,N} \delta(\phi - 2\pi j/N),$$

   Assume the wave function has the form,

$$\psi(\phi) = e^{im\phi} + Be^{-im\phi}, \quad 0 < \phi < 2\pi/N,$$

   where m is not necessarily an integer. Find a transcendental expression for m in terms of β, M, γ and n. Hint: Note the similarity to the Kronig-Penny model, where the solution in Eq. (**??**) translates to this problem with $qa \to m\alpha$, $ka \to \gamma$, and $a \to \alpha$, with $\alpha = 2\pi/N$.

Solution:

a)

$$-\frac{\hbar^2}{2MR^2}\partial_\phi^2\psi + V(\phi)\psi = E\psi.$$

b)

$$e^{iN\gamma} = 1, \quad \gamma = \frac{2\pi j}{N}, \quad j = \text{integer}.$$

c)

$$\begin{aligned}
-\frac{\hbar^2}{2MR^2}\left(\partial_\phi\psi|_{(2\pi/N)+\epsilon} - \partial_\phi\psi|_{(2\pi/N)-\epsilon}\right) + \beta\,\psi|_{2\pi/N} &= 0,\\
\psi &= e^{im\phi} + Be^{-im\phi},\\
\text{Let } \alpha = \frac{2\pi}{N},\ \gamma &= j\alpha,\\
\psi(\alpha) &= \psi(0)e^{ij\alpha},\\
(1)\quad e^{im\alpha} + Be^{-im\alpha} &= e^{ij\alpha}(1+B),\\
-\frac{\hbar^2}{2mR^2}[\psi'(0^+)e^{ij\alpha} - \psi'(\alpha^-)] &= -\beta\psi(\alpha),\\
(2)\quad -im[e^{im\alpha} - Be^{-im\alpha} - e^{ij\alpha}(1-B)] &= p(1+B)e^{ij\alpha},\\
\text{where } p &\equiv \frac{2m\beta R^2}{\hbar^2}
\end{aligned}$$

From (1)

$$B = \frac{e^{im\alpha} - e^{ij\alpha}}{e^{ij\alpha} - e^{-im\alpha}}$$

From (2)

$$B = \frac{(p/m)e^{ij\alpha} + ie^{im\alpha} - ie^{ij\alpha}}{ie^{-im\alpha} - ie^{ij\alpha} - (p/m)e^{ij\alpha}}$$

Equating the two expressions for B,

$$\frac{e^{i(m-j)\alpha} - 1}{1 - e^{-i(m+j)\alpha}} = \frac{(p/m) - i + ie^{i(m-j)\alpha}}{-(p/m) - i + ie^{-i(m+j)\alpha}}$$

Multiply equation by both denominators to get

$$i + (-(p/m) - i)e^{i(m-j)\alpha} + ie^{-2ij\alpha} - ie^{-i(m+j)\alpha} = -i + ie^{i(m+j)\alpha} - ie^{-2ij\alpha} - ((p/m) - i)e^{-i(m+j)\alpha},$$

$$2i - 2ie^{i(m-j)\alpha} + 2ie^{-2ij\alpha} - 2ie^{-i(m+j)\alpha} = \frac{p}{m}\left(-e^{-i(m+j)\alpha} + e^{i(m-j)\alpha}\right),$$

$$2\cos(j\alpha) - 2\cos(m\alpha) = \frac{p}{m}\sin(m\alpha)$$

One needs to solve the last equation for m to find the energy,

$$E = \frac{\hbar^2}{2MR^2}m^2.$$