Chapter 4 – Homework Solutions

1. (a) Show that $\vec{r}^2 = x^2 + y^2 + z^2$ commutes with L_z.

   (b) Show that $\vec{r} \cdot \vec{p}$ commutes with L_z.

Solution:
a)

$$\begin{aligned} L_z &= (-i\hbar)(x\partial_y - y\partial_x), \\ [L_z, x^2 + y^2 + z^2] &= (-i\hbar)\{x\partial_y(x^2 + y^2 + z^2) - y\partial_x(x^2 + y^2 + z^2)\} \\ &= (-i\hbar)(2xy - 2yx) = 0 \;\; \checkmark \end{aligned}$$

b)

$$\begin{aligned} &[L_z, (xp_x + yp_y + zp_z)] \\ &\qquad = (-\hbar)^2\{x\partial_y(x\partial_x + y\partial_y + z\partial_z) - y\partial_x(x\partial_x + y\partial_y + z\partial_z)\} \\ &\qquad + \hbar^2\{x\partial_x(x\partial_y - y\partial_x) + y\partial_y(x\partial_y - y\partial_x) + z\partial_z(x\partial_y - y\partial_x)\} \\ &\qquad = \hbar^2(-x\partial_y + y\partial_x + x\partial_y - y\partial_x) = 0 \;\; \checkmark \end{aligned}$$

2. Any two rotations, $\vec{\alpha}$ and $\vec{\beta}$, can be written as a single rotation by $\vec{\gamma}$, which in the spin 1/2 basis means

$$e^{i\vec{\beta}\cdot\vec{\sigma}/2}e^{i\vec{\alpha}\cdot\vec{\sigma}/2} = e^{i\vec{\gamma}\cdot\vec{\sigma}/2}.$$

Show that the equivalent angle $\vec{\gamma}$ may be written in terms of $\vec{\alpha}$ and $\vec{\beta}$ as

$$\begin{aligned}\cos(\gamma/2) &= \cos(\beta/2)\cos(\alpha/2) - \hat{\beta}\cdot\hat{\alpha}\sin(\alpha/2)\sin(\beta/2)\\ \hat{\gamma}\sin(\gamma/2) &= \cos(\beta/2)\sin(\alpha/2)\hat{\alpha} + \cos(\alpha/2)\sin(\beta/2)\hat{\beta} + \sin(\beta/2)\sin(\alpha/2)\hat{\alpha}\times\hat{\beta},\end{aligned}$$

where $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$ are the corresponding unit vectors. Note that these relations would hold for any rotation, not just the spin 1/2 system. Thus, they describe the rotation group.
Hints: Use the fact that $e^{i\vec{a}\cdot\sigma} = \cos(a) + i\frac{\vec{\sigma}\cdot\vec{a}}{|\vec{a}|}\sin(a)$. Also use the identity $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$.

Solution:

$$\begin{aligned}e^{i\vec{\beta}\cdot\vec{\sigma}/2} &= \cos(\beta/2) + i(\hat{\beta}\cdot\vec{\sigma})\sin(\beta/2),\\ e^{i\vec{\alpha}\cdot\vec{\sigma}/2} &= \cos(\alpha/2) + i(\hat{\alpha}\cdot\vec{\sigma})\sin(\alpha/2),\\ e^{i\vec{\beta}\cdot\vec{\sigma}/2}e^{i\vec{\alpha}\cdot\vec{\sigma}/2} &= \cos(\beta/2)\cos(\alpha/2) + i(\hat{\beta}\cdot\vec{\sigma})\sin(\beta/2)\cos(\alpha/2)\\ &+ i(\hat{\alpha}\cdot\vec{\sigma})\sin(\alpha/2)\cos(\beta/2)\\ &- (\hat{\beta}\cdot\vec{\sigma})(\hat{\alpha}\cdot\vec{\sigma})\sin(\beta/2)\sin(\alpha/2),\\ &= \cos(\gamma/2) + i(\hat{\gamma}\cdot\vec{\sigma})\sin(\gamma/2).\end{aligned}$$

We must find γ for which the last expression is true. First, we need to write the term that is quadratic in sigma matrices as a sum of terms that are linear in sigma matrices or constant,

$$\begin{aligned}\hat{\beta}_i\sigma_i\hat{\alpha}_j\sigma_j &= i\epsilon_{ijk}\sigma_k\hat{\beta}_i\hat{\alpha}_j + \hat{\beta}\cdot\hat{\alpha}\\ &= i(\hat{\beta}\times\hat{\alpha})\cdot\vec{\sigma} + \hat{\beta}\cdot\hat{\alpha}.\end{aligned}$$

After substituting above, one can see that for the terms with no sigma matrices to satisfy the equality

$$\cos(\gamma/2) = \cos(\beta/2)\cos(\alpha/2) - \hat{\beta}\cdot\hat{\alpha}\sin(\alpha/2)\sin(\beta/2).$$

For the terms linear in sigma matrices,

$$\begin{aligned}\sin(\gamma/2)\hat{\gamma} &= \hat{\beta}\sin(\beta/2)\cos(\alpha/2) + \hat{\alpha}\sin(\alpha/2)\cos(\beta/2)\\ &- (\hat{\beta}\times\hat{\alpha})\sin(\beta/2)\sin(\alpha/2).\end{aligned}$$

If you're bored, you can check the expressions for $\cos(\gamma/2)$ and $\sin(\gamma/2)$ to see that they indeed satisfy the identity $\cos^2(\gamma/2) + \sin^2(\gamma/2) = 1$.

3. Consider the matrices,

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad , \quad S_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad , \quad S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

These represent the rotation matrices for angular momentum $S = 1$, $S(S+1) = 2$. Note that the eigenvalues of S_z are -1,0,1 as expected.

(a) Explicitly multiply the matrices to show that

$$[S_i, S_j] = i\hbar\epsilon_{ijk}S_k.$$

For efficiency, just pick one of the three combinations to check.

(b) Explicitly multiply the matrices to show that

$$\sum_i S_i^2 = 2\hbar^2\mathbb{I} = \hbar^2 S(S+1)\mathbb{I}.$$

Solution:
a)

$$S_xS_y - S_yS_x = \frac{\hbar^2}{2}\left\{ \begin{pmatrix} i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & i \end{pmatrix} \right\}$$

$$= i\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = i\hbar S_z \quad \checkmark$$

b)

$$S_x^2 = \frac{\hbar^2}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$S_y^2 = \frac{\hbar^2}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix},$$

$$S_z^2 = \hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$S_x^2 + S_y^2 + S_z^2 = \hbar^2 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2\hbar^2\mathbb{I}.$$

Because these are the matrices for $S = 1$, we expected $\sum_i S_i^2 = S(S+1)$, and indeed for $S(S+1) = 2$ for $S = 1$.

4. Using the definition, $L_z = -i\hbar(x\partial_y - y\partial_x) = -i\hbar\partial_\phi$, express $X(\alpha) = e^{iL_z\alpha/\hbar} x e^{-iL_z\alpha/\hbar}$ in terms of x, y and z. Hint: Note that

$$e^{\alpha\partial_\phi} f(\phi) e^{-\alpha\partial_\phi} = f(\phi + \alpha).$$

because $e^{\alpha\partial_\phi}$ generates a Taylor expansion.

Solution:

$$\begin{aligned}
L_z &= -i\hbar(x\partial_y - y\partial_x) = -i\hbar\partial_\phi, \\
e^{iL_z\alpha/\hbar} &= e^{\alpha\partial_\phi}, \\
e^{iL_z\alpha/\hbar}(r\sin\theta\cos\phi)e^{-iL_z\alpha/\hbar} &= r\sin\theta\cos(\phi+\alpha) \\
&= r\sin(\theta\cos\phi\cos\alpha - r\sin\theta\sin\phi\sin\alpha) \\
&= x\cos\alpha - y\sin\alpha.
\end{aligned}$$

5. Consider the six group elements for the symmetry of the equilateral triangle listed in Sec. **??**. As a six-by-six matrix, find the coefficients a_{ij}.

Solution:

(a) $\mathcal{R}_1$, The identity

(b) $\mathcal{R}_2$, Rotation by 120°

(c) $\mathcal{R}_3$, Rotation by 240°

(d) $\mathcal{R}_4$, Reflecting about an axis through the center of the triangle in the 30° direction

(e) $\mathcal{R}_5$, Reflecting about an axis through the center of the triangle in the 900° direction

(f) $\mathcal{R}_6$, Reflecting about an axis through the center of the triangle in the 150° direction

$$R_i R_j =$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \\ 3 & 1 & 2 & 6 & 4 & 5 \\ 4 & 6 & 5 & 1 & 3 & 2 \\ 5 & 4 & 6 & 2 & 1 & 3 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$$

6. Using the commutation relations for angular momentum, $[L_x, L_y] = i\hbar L_z$, and the definition $L_\pm = L_x \pm iL_y$, show that

$$|\vec{L}|^2 = L_z^2 + L_+L_- - \hbar L_z.$$

Solution:

$$\begin{aligned} L_z^2 + L_+L_- - \hbar L_z &= L_z^2 + (L_x + iL_y)(L_x - iL_y) - \hbar L_z \\ &= L_z^2 + L_x^2 + L_y^2 - i[L_x, L_y] - \hbar L_z \\ &= L_x^2 + L_y^2 + L_z^2 \quad \checkmark \end{aligned}$$

7. In terms of ℓ, m_1 and m_2 find expressions for:

   (a) $\langle \ell m_1 | L_x^2 | \ell m_2 \rangle$ (Warning: this is messy)

   (b) $\langle \ell m_1 | L_x^2 + L_y^2 | \ell m_2 \rangle$

Solution:

$$\begin{aligned} L_+ &= L_x + iL_y, \\ L_- &= L_x - iL_y, \\ L_x &= \frac{1}{2}(L_+ + L_-) \\ L_x^2 &= \frac{1}{4}(L_+^2 + L_-^2 + L_+L_- + L_-L_+) \end{aligned}$$

a)

$$\begin{aligned} L_+|\ell, m_2\rangle &= \hbar\sqrt{\ell(\ell+1) - m_2^2 - m_2}|\ell, m_2+1\rangle, \\ L_+^2|\ell, m_2\rangle &= \sqrt{\ell(\ell+1) - m_2^2 - m_2} \\ &\quad \cdot \sqrt{\ell(\ell+1) - (m_2+1)^2 - (m_2+1)}|\ell, m_2+2\rangle, \\ L_-^2|\ell, m_2\rangle &= \sqrt{\ell(\ell+1) - m_2^2 + m_2} \\ &\quad \cdot \sqrt{\ell(\ell+1) - (m_2-1)^2 - (m_2-1)}|\ell, m_2-2\rangle. \end{aligned}$$

$$\begin{aligned} &\langle \ell, m_1 | L_+^2 | \ell m_2 \rangle \\ &\quad = \{[\ell(\ell+1) - m_2^2 - m_2][\ell(\ell+1) - (m_2+1)^2 - (m_2+1)]\}^{1/2}\delta_{m_2+2,m_1}, \quad (1) \\ &\langle \ell, m_1 | L_-^2 | \ell m_2 \rangle \\ &\quad = \{[\ell(\ell+1) - m_2^2 + m_2][\ell(\ell+1) - (m_2-1)^2 - (m_2-1)]\}^{1/2}\delta_{m_2-2,m_1}\hbar^2, \quad (2), \\ &\quad \langle \ell, m_1 | \frac{1}{2}(|\vec{L}|^2 - L_z^2|\ell, m_2\rangle = \delta_{m_1,m_2}\frac{1}{2}[\ell(\ell+1) - m_1^2]\hbar^2, \quad (3) \\ &\qquad\qquad\qquad \text{answer} = [(1) + (2) + (3)] \end{aligned}$$

b)

$$\begin{aligned} \langle \ell m_1 | (L_x^2 + L_y^2) | \ell m_2 \rangle &= \langle \ell m_1 || \vec{L}|^2 - L_z^2 | \ell m_2 \rangle \\ &= \hbar^2[\ell(\ell+1) - m_1^2]\delta_{m_1,m_2} \end{aligned}$$

8. Consider a particle of mass m in a spherical well of radius R, where the potential is $+\infty$ for $r > R$ and zero for $r < R$.

   (a) Find the ground state energy.

   (b) Describe how one would find the energy of the first excited state of the same well.

   (c) If the particle is an electron and the radius of the well is 0.15 nm, give a numerical value for the energy of the ground state in eV.

Solution:
a) For $\ell = 0$, $u \sim \sin kr$,

$$kR = n\pi, \;\; k_0 = \pi/R,$$
$$E = \frac{\hbar^2\pi^2}{2mR^2}.$$

b) For $\ell = 1$, $u \sim (\sin(kr)/kr - \cos(kr)$.

$$0 = \frac{\sin kr}{kr} - \cos kr,$$
$$\tan(kr) = kr, \quad \text{transcendental eq..}$$

c) Use the fact that $\hbar c = 197.327$ eV nm, and $mc^2 = 0.511 \times 10^6$ eV.

$$E_0 = \frac{(197.327)^2}{2 \cdot 0.511 \times 10^6}\pi^2 \frac{1}{0.15^2}$$
$$= 16.7 \;\; \text{eV}.$$

9. (a) Estimate the ground state binding energies of the following atoms. You can use the fact that the binding energy for hydrogen is 13.6 eV, the mass of an electron is 0.511 MeV the mass of a muon is 105.7 MeV, the mass of a proton is 938.3 MeV and the charge of Pb is 82. Then scale the hydrogen values to get the desired results. The Bohr radius of H is 0.053 nm.

   i. e, Pb
   ii. μ^-, p
   iii. e^+e^-
   iv. $\bar{p}, Pb$

   The mass of a muon is 205 times larger than that of an electron.

   (b) For the same cases above, find the associated Bohr radii. (Treat the Pb nucleus as a point particle)

Solution:
Remember that the energies are proportional to

$$E = -\frac{Ze^2}{2a_0}, \quad a_0 = \frac{\hbar^2}{\mu e^2}.$$

a,i.) $Z = 82$, $E = (-13.6) \cdot 82^2$ eV =-91.4 keV
a,ii.)

$$\begin{aligned} \mu &= \frac{m_\mu m_p}{m_\mu + m_p} = \frac{105.7 \cdot 938.3}{1044} = 95.0 \text{ MeV}, \\ &= 186 m_e, \\ E_0 &= -186 \cdot 13.6 = -2.53 \;\; \text{keV} \end{aligned}$$

a,iii.)

$$\begin{aligned} \mu &= m_e/2, \\ E_0 &= -6.8 \text{ eV}. \end{aligned}$$

a,iv.)

$$\begin{aligned} \mu &= 938 = 1835 m_e, \\ Z &= 82, \\ E_0 &= -1835 \cdot 13.6 \cdot 82^2 = -167 \text{ MeV}. \end{aligned}$$

b,i.)

$$a = \frac{1}{82} a_0 = 0.0645 \text{ nm}.$$

b,ii.)

$$\begin{aligned} \mu &= m_e \cdot 186 \\ a &= a_0/186 = 0.02845 \;\; \text{nm} \end{aligned}$$

b,iii.)

$$a = 2a_0 = 0.106 \text{ nm}$$

b,iv.)

$$\begin{aligned} a &= a_0 \frac{1}{82} \frac{m_e}{m_p} \\ &= 3.5 \times 10^{-16} \text{ m}. \end{aligned}$$

This last number is even smaller than the proton itself – so the idea that the Pb nucleus can be treated as a point particle is rather ridiculous.

10. For the Hydrogen atom, calculate the expectation of the operator X between the ground state and each of the four $n = 2$ states. You can express answer in terms of a_0.

Solution:

$$x = \frac{1}{2}(re^{i\phi} + r^{-i\phi})\sin\theta,$$

By symmetry,

$$\langle n = 1|x|n = 2, \ell = 0\rangle = 0,$$
$$\langle n = 1|x|n = 2, \ell = 1, m = 0\rangle = 0.$$

$$\begin{aligned}
\langle n = 1|re^{-i\phi}\sin\theta|n = 2, \ell = 1, m = 1\rangle &= \int dr\ r^3 R_{n=1}(r)R_{n=2,\ell=1}(r) \\
&\cdot \int d\cos\theta d\phi Y_{\ell=0,m=0}(\theta,\phi)e^{-i\phi}\sin\theta Y_{\ell=1,m=1}(\theta,\phi) \\
&= -\int dr\ r^3 R_{n=1}(r)R_{n=2,\ell=1}(r)\int d\phi\ \sqrt{\frac{3}{8\pi}}\sqrt{\frac{1}{4\pi}}\int d\cos\theta\sin^2\theta \\
&= -\sqrt{\frac{3}{8}}\int dr\ r^3 R_{n=1}(r)R_{n=2,\ell=1}(r).
\end{aligned}$$

From Eq. (4-75)

$$\begin{aligned}
&= -\sqrt{\frac{3}{8}}\frac{2}{a_0^{3/2}}\frac{1}{(2a_0)^{3/2}}\frac{1}{a_0\sqrt{3}}\int dr\ r^4 e^{-r/a_0} \\
&= -\frac{a_0}{4}4!, \\
\langle n = 1|x|n = 2, \ell = 1, m = \pm 1\rangle &= \mp 3a_0.
\end{aligned}$$

11. Prove the following recurrence relation for spherical Bessel functions:

$$j_{\ell+1}(z) = -j'_\ell(z) + \frac{\ell}{z} j_\ell(z).$$

To accomplish this, assume the equation is true and that $j_\ell(z)$ is a solution to:

$$-j''_\ell(z) - \frac{2}{z} j'_\ell(z) + \frac{\ell(\ell+1)}{z^2} j_\ell(z) = j_\ell(z).$$

Then show that using the assumed expression for $j_{\ell+1}(z)$ will be a solution to:

$$-j''_{\ell+1}(z) - \frac{2}{z} j'_{\ell+1}(z) + \frac{(\ell+1)(\ell+2)}{z^2} j_{\ell+1}(z) = j_{\ell+1}(z).$$

This last expression is the same differential equation as the one just above, but with $\ell \to \ell+1$.

Solution:
Take derivative of 1st eq:

$$j'_{\ell+1} = -j''_\ell - \frac{\ell}{z^2} + \frac{\ell}{z} j'_\ell.$$

Using the differential equation for the Bessel function,

$$\begin{aligned} &= \frac{2}{z} j'_\ell - \frac{\ell(\ell+1)}{z^2} j_\ell + j_\ell - \frac{\ell}{z^2} j_\ell + \frac{\ell}{z} j'_\ell \\ &= \frac{2+\ell}{z^2} j'_\ell - \frac{\ell(\ell+2)}{z^2} j_\ell + j_\ell. \end{aligned}$$

Taking another derivative,

$$\begin{aligned} j''_{\ell+1} &= \frac{-(2+\ell)}{z^2} j'_\ell + \frac{2+\ell}{z} j''_\ell + \frac{2\ell(\ell+2)}{z^3} j_\ell - \frac{\ell(\ell+2)}{z^2} j'_\ell + j'_\ell \\ &= \frac{2+\ell}{z} j''_\ell + \left(\frac{-(\ell+1)(\ell+2)}{z^2} + 1 \right) j'_\ell + \frac{2\ell(\ell+2)}{z^3} j_\ell. \quad (a). \end{aligned}$$

We need to show

$$-j''_{\ell+1} - \frac{2}{z} j_{\ell+1} + \frac{(\ell+1)(\ell+2)}{z^2} j_{\ell+1} - j_{\ell+1} =?0.$$

Note that ℓ has been replaced with $\ell+1$. Combining (a) with the expression we are trying to prove,

$$\begin{aligned} 0 =? & \left[-\frac{(\ell+2)}{z} \right] j''_\ell + \left[\frac{(\ell+1)(\ell+2)}{z^2} - 1 - \frac{2(\ell+2)}{z^2} - \frac{(\ell+1)(\ell+2)}{z^2} + 1 \right] j'_\ell \\ & + \left[-\frac{2\ell(\ell+2)}{z^3} + \frac{2\ell(\ell+2)}{z^3} - \frac{2}{z} + \frac{\ell(\ell+1)(\ell+2)}{z^3} - \frac{\ell}{z} \right] j_\ell \\ = & \frac{\ell+2}{z} \left\{ -j'' z - \frac{2}{z} j'_\ell - j_\ell + \frac{\ell(\ell+1)}{z^2} j_\ell \right\}. \end{aligned}$$

The last expression is indeed zero as it is the differential equation for Bessel functions.

12. Find the Clebsch-Gordan coefficient

$$\langle \ell = 1, s = 1, j = 0, m = 0 | \ell = 1, s = 1, m_\ell = 1, m_s = -1\rangle$$

Solution:

$$\begin{aligned}
|j = 2, m = 2\rangle &= |m_\ell = 1, m_s = 1\rangle \\
, J^-|J, m\rangle &= \sqrt{J(J+1) - m^2 + m}|J, m-1\rangle \\
J^-|J = 2, m = 2\rangle &= (L^- + S^-)|m_{\ell=1}, m_s = 1\rangle, \\
\sqrt{6-4+2}|J = 2, m = 1\rangle &= \sqrt{2-1+1}|m_\ell = 0, m_s = 1\rangle + \sqrt{2}|m_\ell = 1, m_s = 0\rangle, \\
|J = 2, m = 1\rangle &= \frac{1}{\sqrt{2}}|m_\ell = 0, m_s = 1\rangle + \frac{1}{\sqrt{2}}|m_\ell = 1, m_s = 0\rangle.
\end{aligned}$$

By orthogonality,

$$|J = 1, m = 1\rangle = \frac{1}{\sqrt{2}}|m_\ell = 0, m_s = 1\rangle - \frac{1}{\sqrt{2}}|m_\ell = 1, m_s = 0\rangle.$$

Now lowering the $m = 2$ states,

$$\begin{aligned}
\sqrt{6}|J = 2, m = 0\rangle &= \frac{1}{\sqrt{2}}\left\{\sqrt{2}|m_\ell = -1, m_s = 1\rangle + \sqrt{2}|m_\ell = 0, m_s = 0\rangle\right\} \\
&\quad + \frac{1}{\sqrt{2}}\left\{\sqrt{2}|m_\ell = 0, m_s = 0\rangle + \sqrt{2}|m_\ell = 1, m_s = -1\rangle\right\}, \\
|J = 2, m = 0\rangle &= \frac{1}{\sqrt{6}}|m_\ell = -1, m_s = 1\rangle + \frac{1}{\sqrt{6}}|m_\ell = 1 m_s = -1\rangle \\
&\quad + \frac{2}{\sqrt{6}}|m_\ell = 0, m_s = 0\rangle.
\end{aligned}$$

Now,lowering the $J = 1$ state,

$$|J = 1, m = 0\rangle = \frac{1}{\sqrt{2}}|m_\ell = -1, m_s = 1\rangle - \frac{1}{\sqrt{2}}|m_\ell = 1, m_s = -1\rangle.$$

Again, by orthogonality,

$$|J = 0, m = 0\rangle = \frac{1}{\sqrt{3}}\left\{|m_\ell = 1, m_s = -1\rangle + |m_\ell = -1, m_s = 1\rangle - |m_\ell = 0, m_s = 0\rangle\right\},$$

$$\langle J = 0, m = 0|m_\ell = 1, m_s = -1\rangle = \frac{1}{\sqrt{3}}.$$

Another way to solve the same problem would be to begin with

$$|J = 0, m = 0\rangle = A|m_\ell = 1, m_s = -1\rangle + B|m_\ell = -1, m_s = 1\rangle + C|m_\ell = 0, m_s = 0\rangle,$$

then solve for the coefficients by requiring $J + |J = 0, m = 0\rangle = 0$.

13. Calculate the Clebsch-Gordan Coefficients $\langle \ell = 12, s = 1, j = 12, m_j = 12 | \ell = 12, s = 1, m_\ell, m_s\rangle$ for all m_ℓ and m_s.

Solution:

$$
\begin{aligned}
|J = 13, m_j = 13\rangle &= |m_\ell = 12, m_s = 1\rangle, \\
\sqrt{13 \cdot 14 - 13^2 + 13}|J = 13, m_j = 12\rangle &= (12 \cdot 13 - 12^2 + 12)^{1/2}|m_\ell = 11, m_s = 1\rangle \\
&\quad + 2^{1/2}|m_\ell - 12, m_s = 0\rangle, \\
|J = 13, m_j = 12\rangle &= \frac{1}{\sqrt{26}}\left\{\sqrt{24}|m_\ell = 11, m_s = 1\rangle + \sqrt{2}|m_\ell = 12, m_s = 0\rangle\right\}.
\end{aligned}
$$

By orthogonality,

$$
\begin{aligned}
&|J = 12, m_j = 12\rangle = \\
&\qquad \frac{1}{\sqrt{26}}\left\{\sqrt{2}|m_\ell = 11, m_s = 1\rangle - \sqrt{24}|m_\ell = 12, m_s = 0\rangle\right\}, \\
&\langle J = 12, m_j = 12|\ell = 12, s = 1, m_\ell = 11, m_s = 1\rangle = \frac{1}{\sqrt{13}}, \\
&\langle J = 12, m_J = 12|\ell = 12, s = 1, m_\ell = 12, m_s = 0\rangle = -\frac{\sqrt{12}}{\sqrt{13}}.
\end{aligned}
$$

All others are zero.

14. An electron is in an $\ell = 1$ state of a hydrogen atom. It experiences a spin orbit interaction,

$$V_{\text{s.o.}} = \alpha \vec{L} \cdot \vec{S}$$

and also feels an external magnetic field

$$V_{\text{B}} = -\mu \vec{B} \cdot \left(\vec{L} + 2\vec{S} \right).$$

a) Using the basis

$$|J = 3/2, M = 3/2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |J = 3/2, M = -3/2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$|J = 3/2, M = 1/2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |J = 1/2, M = 1/2\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|J = 3/2, M = -1/2\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |J = 1/2, M = -1/2\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

write the Hamiltonian components $V_{\text{s.o.}}$ and V_{B} as 6×6 matrices. To assist you, the $|J, M\rangle$ states can be written in the $|m_\ell, m_s\rangle$ basis as

$$\begin{aligned}
|J = 3/2, m_j = 3/2\rangle &= |m_\ell = 1, m_s = 1/2\rangle, \\
|J = 3/2, m_j = -3/2\rangle &= |m_\ell = -1, m_s = -1/2\rangle, \\
\sqrt{15/4 - 9/4 + 3/2}|J = 3/2, m_j = 1/2\rangle &= \sqrt{2}|0, 1/2\rangle + \sqrt{3/4 - 1/4 + 1/2}| - 1, -1/2\rangle, \\
|J = 3/2, m_j = 1/2\rangle &= \sqrt{\frac{2}{3}}|0, 1/2\rangle + \frac{1}{\sqrt{3}}|1, -1/2\rangle, \\
|J = 1/2, m_j = 1/2\rangle &= \frac{1}{\sqrt{3}}|0, 1/2\rangle - \sqrt{\frac{2}{3}}|1, -1/2\rangle, \\
|J = 3/2, m_j = -1/2\rangle &= \sqrt{\frac{2}{3}}|0, -1/2\rangle + \frac{1}{\sqrt{3}}| - 1, 1/2\rangle, \\
|J = 1/2, m_j = -1/2\rangle &= \frac{1}{\sqrt{3}}|0, -1/2\rangle - \sqrt{\frac{2}{3}}| - 1, 1/2\rangle.
\end{aligned}$$

b) What are the six eigenvalues of H?

Solution:
a) In the J, M basis, $V_{\rm s.o.}$ is diagonal. Use the fact that

$$\vec{L}\cdot\vec{S} = \frac{1}{2}(|\vec{J}|^2 - |\vec{L}|^2 - |\vec{S}|^2) = \frac{\hbar^2}{2}\{J(J+1) - L(L+1) - S(S+1)\},$$

which gives

$$\langle J=3/2, M|H_{\rm s.o.}|J=3/2, M\rangle = \frac{\alpha\hbar^2}{2},$$
$$\langle J=1/2, M|H_{\rm s.o.}|J=1/2, M\rangle = -\alpha\hbar^2.$$

$$V_{\rm s.o.} = \frac{\alpha\hbar^2}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

To calculate $V_{\rm B}$ one must calculate the overlap $\langle J, M|V_{\rm B}|J', M'\rangle$ for all 36 combinations of J, M, J', M'. Each state must be expanded in the m_ℓ, m_s basis as given above. Fortunately, there is no overlap unless $M = M'$. Thus, there are only four off-diagonal terms: $\langle J=3/2, M=1/2|V_{\rm B}|J=1/2, M=1/2\rangle$, $\langle J=3/2, M=-1/2|V_{\rm B}|J=1/2, M=-1/2\rangle$ and the Hermitian conjugates.
First, to calculate the six diagonal elements,

$$\begin{aligned}
\langle J=3/2, M=3/2|V_{\rm B}|J=3/2, M=3/2\rangle &= -\mu\hbar B(1+2\cdot 1/2) = -2\mu\hbar B,\\
\langle J=3/2, M=-3/2|V_{\rm B}|J=3/2, M=-3/2\rangle &= 2\mu\hbar B,\\
\langle J=3/2, M=1/2|V_{\rm B}|J=3/2, M=1/2\rangle &= -\mu\hbar B[(2/3)(0+2\cdot 1/2) + (1/3)(1+2\cdot(-1/2))]\\
&= -\frac{2}{3}\mu\hbar B,\\
\langle J=1/2, M=1/2|V_{\rm B}|J=1/2, M=1/2\rangle &= -\mu\hbar B[(1/3)(0+2\cdot 1/2) + (2/3)(1+2\cdot(-1/2))]\\
&= -\frac{1}{3}\mu\hbar B,\\
\langle J=3/2, M=-1/2|V_{\rm B}|J=3/2, M=-1/2\rangle &= \frac{2}{3}\mu\hbar B,\\
\langle J=1/2, M=-1/2|V_{\rm B}|J=1/2, M=-1/2\rangle &= \frac{1}{3}\mu\hbar B.
\end{aligned}$$

The off-diagonal elements are

$$\begin{aligned}
\langle J=3/2, M=1/2|V_{\rm B}|J=1/2, M=1/2\rangle &= \langle J=1/2, M=1/2|V_{\rm B}|J=3/2, M=1/2\rangle = -\mu\hbar B\{\frac{\sqrt{}}{3}\\
&= \frac{\sqrt{2}}{3}\mu\hbar B,\\
\langle J=3/2, M=-1/2|V_{\rm B}|J=1/2, M=-1/2\rangle &= \langle J=1/2, M=-1/2|V_{\rm B}|J=3/2, M=-1/2\rangle\\
&= -\frac{\sqrt{2}}{3}\mu\hbar B,
\end{aligned}$$

In matrix form,

$$V_{\mathrm{B}} = \mu\hbar B \begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2/3 & \sqrt{2}/3 & 0 & 0 \\ 0 & 0 & \sqrt{2}/3 & -1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2/3 & -\sqrt{2}/3 \\ 0 & 0 & 0 & 0 & -\sqrt{2}/3 & 1/3 \end{pmatrix}$$

b) The first two eigenvalues can be read off the matrices

$$E_1 = \frac{\alpha\hbar^2}{2} - 2\mu\hbar B,$$
$$E_2 = \frac{\alpha\hbar^2}{2} + 2\mu\hbar B.$$

The last four eigenvalues can be found by realizing that the remaining 6×6 matrix can be reduced to two 2×2 submatrices. The first 2×2 submatrix is

$$H_{3,4} = \begin{pmatrix} \alpha\hbar^2/2 - (2/3)\mu\hbar B & (\sqrt{2}/3)\mu\hbar B \\ (\sqrt{2}/3)\mu\hbar B & -\alpha\hbar^2 - (1/3)\mu\hbar B \end{pmatrix}$$
$$= (-\alpha^2/4 - \mu\hbar B/2)\mathbb{I} + (3\alpha^2/4 - \mu\hbar B/6)\sigma_z + \mu\hbar B\frac{\sqrt{2}}{3}\sigma_x.$$

The eigenvalues are then

$$E_{3,4} = -\alpha^2/4 - \mu\hbar B/2 \pm \sqrt{(3\alpha^2/4 - \mu\hbar B/6)^2 + (2/9)(\mu\hbar B)^2}.$$

To obtain the last two eigenvalues, one first writes down the last submatrix,

$$H_{5,6} = \begin{pmatrix} \alpha\hbar^2/2 + (2/3)\mu\hbar B & -(\sqrt{2}/3)\mu\hbar B \\ -(\sqrt{2}/3)\mu\hbar B & -\alpha\hbar^2 + (1/3)\mu\hbar B \end{pmatrix}$$
$$= (-\alpha^2/4 + \mu\hbar B/2)\mathbb{I} + (3\alpha^2/4 + \mu\hbar B/6)\sigma_z - \mu\hbar B\frac{\sqrt{2}}{3}\sigma_x.$$

The eigenvalues are

$$E_{5,6} = -\alpha^2/4 + \mu\hbar B/2 \pm \sqrt{(3\alpha^2/4 + \mu\hbar B/6)^2 + (2/9)(\mu\hbar B)^2}.$$

15. A spin 1/2 particle is bound to a fixed center by a spherically symmetric potential. The particle is in an $\ell = 0$ state with spin-up, i.e.

$$\Psi(\vec{r}, m) = \psi(r) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

In terms of $\psi(r)$ and $\vec{r}$, write the matrix element for

$$\langle \vec{r}, m_s | \vec{\sigma} \cdot \vec{r} | \Psi \rangle$$

(a) for $m_s = 1/2$

(b) for $m_s = -1/2$

Solution:

a)

$$\vec{\sigma} \cdot \vec{r} = x\sigma_x + y\sigma_y + z\sigma_z.$$

For the matrix element, the spin and spatial parts factorize,

$$\langle \vec{r}, m_s | \vec{\sigma} \cdot \vec{r} | \Psi \rangle = \chi_s^\dagger \vec{\sigma} \chi_\uparrow \cdot \langle \vec{r} | \psi \rangle.$$

For spin-up, only the σ_z term contributes.

$$\begin{aligned} \langle \vec{r}, m_s = 1/2 | \vec{r} \cdot \vec{\sigma} | \Psi \rangle &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}^\dagger \sigma_x \begin{pmatrix} 1 \\ 0 \end{pmatrix} \langle \vec{r} | z | \psi \rangle \\ &= \psi(r) z = \psi(r) r \cos\theta. \end{aligned}$$

b) For spin-down only the σ_x and σ_y terms contribute

$$\begin{aligned} \langle \vec{r}, m_s = -1/2 | \vec{r} \cdot \vec{\sigma} | \Psi \rangle &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}^\dagger \sigma_x \begin{pmatrix} 1 \\ 0 \end{pmatrix} \langle \vec{r} | x | \psi \rangle \\ &+ \begin{pmatrix} 0 \\ 1 \end{pmatrix}^\dagger \sigma_y \begin{pmatrix} 1 \\ 0 \end{pmatrix} \langle \vec{r} | y | \psi \rangle \\ &= x\psi(r) + iy\psi(r) = \psi(r) r \sin\theta e^{i\phi} \end{aligned}$$