# Chapter 1 – Homework Solutions

1. Photons, traveling along the $z$ axis can be polarized either linearly along the $x$ or $y$ axis, or a linear combination of the two states.

   (a) Write the operator that rotate states by 45° about the $z$ axis in terms of $|x\rangle$, $|y\rangle$ and the corresponding bras.

   **Solution**:
   A rotation that rotates by 45° changes the states $|x\rangle$ and $|y\rangle$ as

   $$\begin{aligned} |x\rangle &\to (|x\rangle + |y\rangle)/\sqrt{2}, \\ |y\rangle &\to (|y\rangle - |x\rangle)/\sqrt{2}. \end{aligned}$$

   The operator that performs the translation is, by inspection,

   $$\begin{aligned} \mathcal{R} = &\frac{1}{\sqrt{2}}(|x\rangle + |y\rangle)\langle x| \\ &+ \frac{1}{\sqrt{2}}(|y\rangle - |x\rangle)\langle y|. \end{aligned}$$

   Or

   $$\mathcal{R} = \frac{1}{\sqrt{2}}\left(|x\rangle\langle x| + |y\rangle\langle x| - |x\rangle\langle y| + |y\rangle\langle y|\right).$$

(b) Choosing the basis,

$$|x\rangle \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} \; , \quad |y\rangle \to \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

write the matrix that rotates the states by $\phi$ about the $z$ axis.

**Solution**:
From the previous problem one can see that the answer is

$$\mathcal{R} \; = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}. \tag{0.1}$$

(c) Right-hand circularly polarized (RCP) light is made of a linear combination of $x$ and $y$ polarized light.

$$|R\rangle = \frac{1}{\sqrt{2}} \left( |x\rangle + i|y\rangle \right).$$

Light traveling along the $z$ axis passes through a thin slab of thickness $Z$ whose index of refraction, $k = n\omega/c$, is different for light polarized in the $x$ and $y$ directions. In terms of $n_x$, $n_y$ and $Z$ find the polarization vector for light which enters the slab as right-circularly polarized.
HINT: The wave has a form $e^{-i\omega t + ikz}$. The two components have the same $\omega$ but different $k$ while in the medium.

**Solution**:

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left( e^{ik_x Z - i\omega t}|x\rangle + i e^{ik_y Z - i\omega t}|y\rangle \right).$$

The indices of refraction give

$$k_x = n_x \omega/c, \;\; k_y = n_y \omega/c.$$

Thus,

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left( e^{-i\omega(t - n_x Z/c)}|x\rangle + i e^{-i\omega(t - n_y Z/c)}|y\rangle \right).$$

Factoring the phase,

$$|\Psi(Z)\rangle \sim (|x\rangle + i e^{i\omega(n_y - n_x)Z/c}|y\rangle)/\sqrt{2}.$$

(d) Find the density matrix for right-circularly polarized light in the basis defined above.

**Solution**:

$$\begin{aligned}\rho &= |\Psi(Z)\rangle\langle\Psi(Z)| \\ &= \left(|x\rangle\langle x| + i|y\rangle\langle x| - i|x\rangle\langle y| + |y\rangle\langle y|\right)/2 \\ &= \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}.\end{aligned}$$

(e) Using the basis described above, write the density matrix for light that is an incoherent mixture, 50% polarized along the $x$ direction and 50% along the $y$ direction.

**Solution**:
Average the density matrix for LCP and for RCP

$$\begin{aligned}\rho &= \frac{1}{4}\left(|x\rangle\langle x| + i|y\rangle\langle x| - i|x\rangle\langle y| + |y\rangle\langle y|\right)\\ &= \frac{1}{4}\left(|x\rangle\langle x| - i|y\rangle\langle x| + i|x\rangle\langle y| + |y\rangle\langle y|\right)\\ &= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.\end{aligned}$$

2. Considering a photon's polarization, calculate $\langle x|\mathcal{R}(\phi)|x\rangle$ for $\phi = \pi/2, \pi, 2\pi$, where the rotation is about the $z$ axis.

**Solution**:
The states $|x\rangle$ and $|y\rangle$ rotate amongst one another like the $\hat{x}$ and $\hat{y}$,

$$|x'\rangle = \cos\phi|x\rangle + \sin\phi|y\rangle,$$
$$|y'\rangle = \cos\phi|y\rangle - \sin\phi|x\rangle.$$

The overlap of $|x'\rangle$ with the original state is

$$\begin{aligned}\langle x'|x\rangle &= \cos\phi \\ &= \begin{cases} 0, & \phi = \pi/2, \\ -1, & \phi = \pi, \\ 1, & \phi = 2\pi. \end{cases}\end{aligned}$$

3. For a spin 1/2 particle, calculate $\langle z,+|\mathcal{R}(\theta)|z,+\rangle$, for the same angles, $\theta = \pi/2, \pi, 2\pi$, when the rotation is about the $y$ axis.

**Solution**:
The rotation operator is

$$\begin{aligned}\mathcal{R}(\theta) &= e^{-i\vec{S}\cdot\vec{\theta}/\hbar} \\ &= e^{-i\sigma_y\theta/2} \\ &= \cos(\theta/2) - i\sigma_y \sin(\theta/2).\end{aligned}$$

Given that the original state is

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and that

$$\sigma_y \begin{pmatrix} 1 \\ 0 \end{pmatrix} = i \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

the overlap of the rotate state with the original state is

$$\begin{aligned}\begin{pmatrix} 1 \\ 0 \end{pmatrix}^\dagger \mathcal{R}(\theta) \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \cos(\theta/2) \\ &= \begin{cases} 1/\sqrt{2}, & \theta = \pi/2, \\ 0, & \theta = \pi \\ -1, & \theta = 2\pi. \end{cases}\end{aligned}$$

Note that you rotate by $2\pi$ but end up with a different (only by a sign) state.

4. Show that the unit matrix $\mathbb{I}$, which can be considered as an operator, is unchanged by a unitary transformation. Begin with the fact that for any matrix $\mathcal{M}$, $\mathcal{M}\mathbb{I} = \mathbb{I}\mathcal{M} = \mathcal{M}$.

**Solution**:

$$U\mathbb{I}U^\dagger = UU^{-1}\mathbb{I} = \mathbb{I}.$$

5. Consider the rotation matrix for rotating Pauli spinors by an angle 90° about the $z$ axis. Using Eq. (**??**),

$$U = e^{-i\sigma_z \pi/4} = \frac{1}{\sqrt{2}}(1 - i\sigma_z).$$

(a) Using the commutator and anti-commutator relations for the $\sigma$ matrices, show that the transformation of $\sigma_x$ is

$$U\sigma_x U^\dagger = \sigma_y.$$

**Solution**:

$$\begin{aligned} U\sigma_x U^\dagger &= \frac{1}{2}\left[(1 - i\sigma_z)\sigma_x(1 + i\sigma_z)\right] \\ &= \frac{1}{2}\left[\sigma_x - i\sigma_z\sigma_x + i\sigma_x\sigma_z + \sigma_z\sigma_x\sigma_z\right] \\ &= \frac{1}{2}\left[\sigma_x + \sigma_y + \sigma_y + i\sigma_y\sigma_z\right] \\ &= \frac{1}{2}\left[\sigma_x + 2\sigma_y - \sigma_x\right] = \sigma_y. \end{aligned}$$

(b) Show that rotating the state, $|+, x\rangle$, which refers to an eigenstate of $\sigma_x$ with eigenvalue of $+1$, gives

$$U|+, x\rangle = |+, y\rangle,$$

which is the eigenstate of $\sigma_y$ with eigenvalue $+1$.

**Solution**:

$$\begin{aligned} \sigma_y U|+, x\rangle &= \frac{1}{\sqrt{2}}\sigma_y(1 - i\sigma_z)|+, x\rangle \\ &= \frac{1}{\sqrt{2}}(\sigma_y + \sigma_x)|+, x\rangle \\ &= \frac{1}{\sqrt{2}}(1 + \sigma_y)|+, x\rangle \\ &= \frac{1}{\sqrt{2}}(1 + \sigma_y\sigma_x)|+, x\rangle \\ &= \frac{1}{\sqrt{2}}(1 - i\sigma_z)|+, x\rangle \\ &= U|+, x\rangle. \end{aligned}$$

6. Consider some Hermitian $N \times N$ matrix $K_{ij}$, with eigenvalues $\lambda^{(n)}$ and the corresponding normalized eigenvectors $v^{(n)}$,

$$Kv^{(n)} = \lambda^{(n)} v^{(n)}.$$

The $N$ eigenvectors each have $N$ components, $v_i^{(n)}$. Create an $N \times N$ matrix

$$U_{ij} = v_j^{*(i)}.$$

Thus, one is making a matrix by having each row be one of the eigenvectors.

(a) Show that $U$ is unitary.

**Solution**:

$$\begin{aligned} U^\dagger_{ji} &= v_j^{(i)} \\ U_{ij} U^\dagger_{jk} &= v_j^{(i)*} v_j^{(k)} = \delta_{ik}. \end{aligned}$$

(b) Show that the $j^{\text{th}}$ component of the vector $Uv^{(n)}$ is

$$(Uv^{(n)})_j = \delta_{nj},$$

Thus, the vectors $Uv^{(n)}$ are

$$(Uv^{(1)}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (Uv^{(2)}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots$$

Now, consider the matrix

$$K' = (UKU^\dagger),$$

and have it act on the vectors above. Show that

$$K'(Uv^{(n)}) = \lambda_n (Uv^{(n)}).$$

This shows that the vectors $(Uv^{(n)})$ are eigenvectors of the matrix $K'$ with eigenvalues $\lambda_n$. Given that the eigenvectors are of the simple form above, the matrix $(UKU^\dagger)$ must be diagonal. Thus, the matrix $U$ defined above provides the unitary matrix for transforming the matrix $K$ into its diagonal form.

**Solution**:

$$\begin{aligned} (Uv^{(n)})_j &= U_{ji} v_i = v_i^{*(j)} v_i^{(n)} = \delta_{jn}. \\ K' &= UKU^\dagger \\ K'Uv^{(n)} &= UU^\dagger K' Uv^{(n)} = UKv^{(n)} = \lambda^{(n)} Uv^{(n)}. \end{aligned}$$

7. Consider the matrix:

$$\mathcal{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(a) What are the eigenvalues of $\mathcal{M}$?

**Solution**:

$$\mathcal{M} = \begin{pmatrix} 1 & 0 \quad 0 \\ 0 & \\ 0 & (\sigma_x) \end{pmatrix}.$$

You can rotate $\sigma_x$ into $\sigma_z$, so

$$\mathcal{M}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The eigenvalues are $1, 1, -1$.

(b) Find eigenvectors of $\mathcal{M}$?

**Solution**:
Because the eigen vectors of $\sigma_x$ are

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

by inspection, the eigenvectors are

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \text{ and } \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

8. Consider the 2×2 matrix

$$\mathcal{K} = \begin{pmatrix} A & C^* \\ C & B \end{pmatrix}$$

(a) What are the eigenvalues of $\mathcal{K}$?

**Solution**:
By inspection, one can rewrite $\mathcal{K}$ as

$$\mathcal{K} = \frac{A+B}{2}\mathbb{I} + \frac{A+B}{2}\sigma_z + C_R\sigma_x + C_I\sigma_y.$$

Because the sigma matrices rotate like vectors, diagonalizing them should lead to something times $\sigma_z$, with the something being the magnitude of the vector representing the coefficients,

$$\lambda_\pm = \frac{A+B}{2} + \sqrt{\left(\frac{A+B}{2}\right)^2 + C_R^2 + C_I^2}.$$

(b) What are the eigenvectors of $\mathcal{K}$?

**Solution**:

$$\begin{pmatrix} A & C^* \\ C & B \end{pmatrix}\begin{pmatrix} 1 \\ u_\pm \end{pmatrix} = \begin{pmatrix} A + C^* u_\pm \\ C + Bu_\pm \end{pmatrix} \qquad = \begin{pmatrix} \lambda_\pm \\ \lambda_\pm u_\pm \end{pmatrix}.$$

$$A + C^* u_\pm = \lambda_\pm,$$

$$u_\pm = \frac{\lambda_\pm - A}{C^*}.$$

The normalized eigenvectors are:

$$\begin{pmatrix} 1 \\ \frac{\lambda_\pm - A}{C^*} \end{pmatrix} \frac{1}{\sqrt{1 + (\lambda_\pm - A)^2/|C|^2}}.$$

9. A beam of light with wavelength 660 $nm$ is sent along the z axis through a polaroid filter that passes only x polarized light. The beam is initially polarized at 30° to the x axis, and the total energy of the pulse is exactly 10 Joules. Estimate the fluctuations of the energy of the transmitted beam, $\langle (E-\bar{E})^2\rangle^{1/2}$. Express the fluctuations as a fraction of the average transmitted energy. (Hint: Consider the binomial distribution, with $N$ tries with probability $p$ of success of each try.)

**Solution**:

$$
\begin{aligned}
u &= \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}, \\
|\langle u|x\rangle|^2 &= p_x(\text{probability of passing through}) \\
N &= \frac{E_{\rm tot}}{\hbar c/\lambda} = \text{number of tries}, \\
P(n) &= \frac{p_x^n(1-p_x)^{N-n}N!}{n!(N-n)!}, \\
\langle n\rangle &= \sum_n P(n)n = \sum_n \frac{p_x^{n-1}(1-p_x)^{N-1-(n-1)}}{(N-1-(n-1))(n-1)!}(N-1)!p_xN \\
&= p_xN \\
\langle n(n-1)\rangle &= p_x^2N(N-1), \\
\text{Fluctuation} &= \langle (n-\langle n\rangle)^2\rangle \\
&= \langle n^2 > -\langle n\rangle^2 = \langle n(n-1)\rangle + \langle n\rangle - \langle n\rangle^2 \\
&= p_x^2(N(N-1)) + p_xN - p_x^2N^2 \\
&= p_x(1-p_x)N. \\
\langle (E-\bar{E})^2\rangle^{1/2} &= \frac{\hbar c}{\lambda}[p_x(1-p_x)N]^{1/2} \\
&= \sqrt{E_{\rm tot}\frac{\hbar c}{\lambda}}\sqrt{p_x(1-p_x)}.
\end{aligned}
$$

10. Consider light moving along the $z$ axis and using the following definitions for $|R\rangle$ and $|L\rangle$ in terms of $x$ and $y$ polarized light,

$$|R\rangle \equiv \frac{1}{\sqrt{2}}(|x\rangle + i|y\rangle), \quad |L\rangle \equiv \frac{1}{\sqrt{2}}(|x\rangle - i|y\rangle),$$

(a) In terms of $|R\rangle$(RCP) and $|L\rangle$(LCP) write the states $|45\rangle$ and $|135\rangle$ which are linearly polarized at 45° and 135° relative to the $x$ axis.

(b) Calculate the $2 \times 2$ transformation matrix from the $45, 135$ basis, where

$$|45\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ |135\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

to the $R, L$ basis.

(c) Show that this transformation is unitary.

**Solution**:

a)

$$\begin{aligned} |45\rangle &= \frac{1}{\sqrt{2}}(|x\rangle + |y\rangle) \\ &= \frac{(1-i)}{2}|R\rangle + \frac{(1+i)}{2}|l\rangle, \\ |135\rangle &= -\frac{(1+i)}{2}|R\rangle - \frac{(1-i)}{2}|L\rangle, \end{aligned}$$

b)

$$\begin{aligned} U &= \begin{pmatrix} \langle 45|R\rangle & \langle 135|R\rangle \\ \langle 45|L\rangle & \langle 135|L\rangle \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} (1+i) & (-1+i) \\ 1-i & -1-i \end{pmatrix} \end{aligned}$$

c)

$$\begin{aligned} UU^\dagger &= \frac{1}{4}\begin{pmatrix} (1+i) & (-1+i) \\ 1-i & -1-i \end{pmatrix}\begin{pmatrix} (1-i) & (1+i) \\ -1-i & -1_i \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} (2+2) & (2-2) \\ (2-2) & (2+2) \end{pmatrix} \\ &= \mathbb{I}. \end{aligned}$$

11. The probability that a photon in state $|\Psi\rangle$ passes through an x-polaroid is the average value of a physical observable which might be called the *x-polarizedness.*.

(a) Write down the operator $P_x$, as a matrix in the $XY$ basis where

$$|X\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ |Y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The projection $\langle\Psi|P_x|\Psi\rangle$ is the probability that $|\Psi\rangle$ makes it through the filter.

(b) What are its eigenvalues and eigenstates?

(c) Write the matrix $P_x$ in the $RL$ basis, where RCP and LCP states are

$$|R\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ |L\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and show that the eigenvalues are the same as in the $XY$ basis. Also, show that the this matrix is a projection operator by explicitly multiplying $P_x$ by itself.

**Solution**:
a)

$$P_x = |x\rangle\langle x| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

b) Eigenvalues are

$$\lambda = 1, 0$$

Eigenvectors are

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

c)

$$|x\rangle = \frac{1}{\sqrt{2}}(|R\rangle + |L\rangle)$$
$$|x\rangle\langle x| = \frac{1}{2}|R\rangle\langle R| + \frac{1}{2}|R\rangle\langle L| + \frac{1}{2}|L\rangle\langle R| + \frac{1}{2}|L\rangle\langle L|$$

In $RL$ basis

$$|R\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$
$$|L\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

So,

$$|x\rangle\langle x| = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$
$$= \frac{1}{2}(1 + \sigma_x).$$

If you rotate this the $\sigma_x$ can turn into $\sigma_z$ which diagonalizes it, so the eigenvalues are $1, 0 \quad \checkmark$.

12. The trace of a matrix $A$ is defined as:

$$\mathrm{Tr}A \equiv \sum_i A_{ii}$$

(a) Show that the trace of $A$ is invariant under a transformation of basis,

$$A \to U^\dagger A U$$

(b) Show that $\mathrm{Tr}AB = \mathrm{Tr}BA$.

**Solution**:

a)

$$\begin{aligned}\mathrm{Tr}U^\dagger A U &= U^\dagger_{ij} A_{jk} U_{ki} \\ &= U_{ki} U^\dagger_{ij} A_{jk} = A_{kk} = \mathrm{Tr}A. \quad \checkmark\end{aligned}$$

b)

$$\mathrm{Tr}AB = A_{ij}B_{ji} = B_{ji}A_{ij} = \mathrm{Tr}BA. \quad \checkmark$$

13. A plane polarized photon at $\theta = 45°$ enters a special crystal with indices of refraction:
$n_x$=1.50 for photons polarized along the $x$ axis
$n_y$=1.52 for photons polarized along the $y$ axis.
Assuming the wavelength of the light is 660 $nm$ before it enters the crystal, choose the thickness of the crystal such that the outgoing light is right circularly polarized. Assume the dispersion is linear, $k = n\omega/c$.

**Solution**:

$$\begin{aligned} u &= \frac{e^{-i\omega\tau}}{\sqrt{2}} \begin{pmatrix} e^{ik_x t} \\ e^{ik_y t} \end{pmatrix} \\ &= \frac{e^{ik_x t - i\omega\tau}}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{i\Delta n\omega t/c} \end{pmatrix} \end{aligned}$$

For RCP $e^{i\Delta n\omega t/c} = i$,

$$\begin{aligned} \frac{\pi}{2} &= \Delta n\omega t/c \\ t &= \frac{\pi c}{2\omega\Delta n}. \end{aligned}$$

14. Consider the matrix for rotation about the $z$ axis,

$$R(\phi) = e^{-i\sigma_z \phi/2}. \tag{0.2}$$

Show that after rotation about the $z$ axis,

$$R(\phi)\sigma_x R^{-1}(\phi) = \sigma_x \cos(\phi) + \sigma_y \sin(\phi) \tag{0.3}$$

**Solution**:

$$\begin{aligned}
\mathcal{R}\sigma_x\mathcal{R}^{-1} &= [\cos(\phi/2) - i\sigma_z \sin(\phi/2)]\,\sigma_x\,[\cos(\phi/2) + i\sigma_z \sin(\phi/2)] \\
&= \sigma_x \cos^2(\phi/2) + \sigma_z\sigma_x\sigma_z \sin^2(\phi/2) - [\sigma_z, \sigma_x] \sin(\phi/2)\cos(\phi/2). \\
\sigma_z\sigma_x\sigma_z &= -\sigma_x, \quad [\sigma_z, \sigma_x] = 2i\sigma_y, \\
\mathcal{R}\sigma_x\mathcal{R}^{-1} &= \sigma_x \cos\phi + \sigma_y \sin(\phi).
\end{aligned}$$

15. Consider a basis for spin-up and spin-down electrons (along the $z$ axis),

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(a) Write down the 4 vectors describing an electron with spin pointed along the positive/negative directions of $x$ and $y$ axes.

(b) Write the six density matrices describing electrons polarized along the positive/negative directions of each of the three axes.

(c) Write the density matrix describing an incoherent mixture of 60% spin-up and 40% spin down.

(d) Using the density matrix, calculate $< y,+|S_z|y,+\rangle$.

**Solution**:

a)

$$|x\uparrow\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$
$$|x\downarrow\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$
$$|y\uparrow\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix},$$
$$|x\uparrow\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix},$$

b)

$$\rho_{z\uparrow} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$
$$\rho_{z\downarrow} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\rho_{x\uparrow} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$
$$\rho_{x\downarrow} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$
$$\rho_{y\uparrow} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix},$$
$$\rho_{y\uparrow} = \frac{1}{2}\begin{pmatrix} 1 & i \\ 0 & -i \end{pmatrix}.$$

c)

$$P_{60/40} = \begin{pmatrix} 0.6 & 0 \\ 0 & 0.4 \end{pmatrix}.$$

d)

$$
\begin{aligned}
\langle y\uparrow|S_z|y\downarrow\rangle &= \frac{1}{2}\mathrm{Tr}\rho_{y\uparrow}\sigma_z \\
&= \frac{1}{2}\mathrm{Tr}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= 0.
\end{aligned}
$$

16. Neutral Kaon Oscillations: There are two kinds of neutral kaons one can make using down and strange quarks,

$$|K^0\rangle = |d\bar{s}\rangle, \quad |\bar{K}^0\rangle = |s\bar{d}\rangle.$$

If it weren't for the weak interaction, the two species would have equal masses, and the Hamiltonian (for a kaon with zero momentum) would be

$$H_0 = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}.$$

However, there is an additional term from the weak interaction that mixes the states,

$$H_m = \begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix}.$$

The masses of a neutral kaon are 497.6 MeV, without mixing, but after adding the mixing term the masses differ by 3.56$\mu$eV. The two eigenstates are known as $K_S$ (K-short) and $K_L$ (K-long), because they decay with quite different lifetimes.

(a) What is $\epsilon$?

(b) If one creates a kaon in the $K_0$ state at time $t = 0$, find the probability it would be measured as a $\bar{K}^0$ as a function of time.

(c) A beam kaons is created in the $K_0$ channel and has a kinetic energy of 600 MeV per kaon. Plot the probability that the kaon is in the $K_0$ state as a function of the distance traveled, $x$. Ignore the fact that the kaons decay.

(d) Repeat (c), but take into account the decays. The states

$$|K_S\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle + |\bar{K}^0\rangle,$$
$$|K_L\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle - |\bar{K}^0\rangle,$$

known at $K$-short and $K$-long, represent the eigenstates of the Hamiltonian. The lifetime of a $K_L$ is 51.2 ns, and the lifetime of the $K_S$ is 0.0896 ns. Note that the wave function should be modified by the factor $e^{-t/(2\tau)}$ to take decays into account decays of lifetime $\tau$. It is often convenient to remember that $\hbar c$=197.327 MeV fm = 197.327 eV nm.

FYI: If the above were exactly true, the $K_S$ state would be even under CP while the $K_L$ would be odd under $CP$. Here, CP is an operator that changes particles to anti-particles. If the particle-antiparticle symmetry were exact, the CP operator would commute with the Hamiltonian and the eigenstates of the Hamiltonian, $K_S$ and $K_L$, would have to be eigenstates of $CP$. The $K_S$ would then decay to two pions and the $K_L$ could decay to three pions. However, there is an additional small CP violating term in the Hamiltonian which allows $K_L$ to have a small probability of decaying to two pions. This was the first experimental laboratory observation of CP violation. CP violation is required to explain the preponderance of matter vs. anti-matter in the universe.

**Solution**:
a)

$$\begin{aligned} H &= M\mathbb{I} + \epsilon\sigma_x, \\ E_\pm &= M \pm \epsilon, \quad \epsilon = 1.78\ \mu\text{eV}. \end{aligned}$$

b)

$$\begin{aligned} |K_0\rangle &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\bar{K}_0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ |\psi(t)\rangle &= e^{-iHt/\hbar}|\psi(0)\rangle = e^{-iMt/\hbar}e^{-i\epsilon\sigma_x t/\hbar}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= e^{-iMt/\hbar}\left[\cos(\epsilon t/\hbar) - i\sin(\epsilon t/\hbar)\right]\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= e^{-iMt/\hbar}\begin{pmatrix} \cos(\epsilon t/\hbar) \\ -i\sin(\epsilon t/\hbar) \end{pmatrix}, \\ P_{K_0} &= \cos^2(\epsilon t/\hbar). \end{aligned}$$

c) The time in rest frame is

$$\begin{aligned} \tau &= z/(\gamma v), \\ \gamma &= E/m = (600 + 497.6)/497.6 = 2.206, \quad \gamma v = \sqrt{\gamma^2 - 1} = 1.966, \\ P_{K_0} &= \cos^2\left(\frac{\epsilon z}{\hbar\gamma v}\right) = \cos^2\left(\frac{\epsilon z}{1.966\hbar c}\right), \\ \hbar c &= 197.326\text{eV nm}, \\ P_{K_0} &= \cos^2(0.004588z), \qquad z \text{ is in nm} \\ &= \cos^2(4.588z), \qquad z \text{ is in } \mu\text{m} \end{aligned}$$

d) Begin with the fact that

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}(|K_S\rangle + |K_L\rangle).$$

Note that $K_S$ and $K_L$ are the energy eigenstates with eigenvalues $M \pm \epsilon/2$. The wave functions evolve as

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}e^{iMt/\hbar}\left(e^{i\epsilon t/2\hbar - t/2\tau_S}|K_S\rangle + e^{+i\epsilon t/2\hbar - t/2\tau_L}|K_L\rangle\right).$$

Rewriting the terms with $|K_S\rangle$ and $|K_L\rangle$ in terms of $|K_0\rangle$ and $|\bar{K}_0\rangle$,

$$|\psi(t)\rangle = \frac{1}{2}e^{iMt/\hbar}\left(e^{i\epsilon t/2\hbar - t/2\tau_S} + e^{-i\epsilon t/2\hbar - t/2\tau_S}\right)|K_0\rangle + \cdots|\bar{K}_0\rangle.$$

There is no need to write out the $|\bar{K}_0\rangle$ term. The probability is then

$$P_{K_0}(t) = \frac{1}{4}e^{-t/\tau_S} + \frac{1}{4}e^{-t/\tau_L} + \frac{1}{2}e^{-t/2\tau_S - t/2\tau_L}\cos(\epsilon t/\hbar).$$

As a check, one can set the lifetimes to infinity and recover the previous expressions. To obtain the answer in terms of $z$, perform the same substitution as in (c), $\epsilon t/\hbar = 4.588z$, with $z$ in $\mu$m. Unless the resolution for measuring decays is of $\mu$m precision, the oscillating term can be neglected. The decays then proceed as one would expect – half the decays with the longer lifetime and half with the longer lifetime. If one goes more than a meter downstream, the beam becomes pretty much perfectly $K_L$.

17. Neutrino Oscillations: There are three kinds of neutrinos corresponding to the three lepton families, and recent evidence has suggested that they may oscillate between generations. Here we consider two flavors, the $\mu$ neutrino and the $\tau$ neutrino. Suppose that the Hamiltonian can be written as a free term plus a term that mixes the $\mu$ and $\tau$ neutrinos, which is proportional to $\alpha$.

$$\mathcal{H} = \begin{pmatrix} \sqrt{k^2 + m_\mu^2} & 0 \\ 0 & \sqrt{k^2 + m_\tau^2} \end{pmatrix} + \alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(a) Supposing you are in the rest frame of the neutrino and that the momentum $k$ is zero, show that the evolution operator $e^{-i\mathcal{H}t/\hbar}$ can be written as

$$e^{-i(m_\mu + m_\tau)t/2\hbar} \{\cos \omega t - i\sigma_n \sin \omega t\},$$

where

$$\hbar\omega \equiv \sqrt{\alpha^2 + (\frac{m_\tau - m_\mu}{2})^2}$$

$$\sigma_n \equiv \frac{\frac{m_\mu - m_\tau}{2}\sigma_z + \alpha\sigma_x}{\hbar\omega}$$

(b) If a neutrino starts as a $\mu$ neutrino, what is the probability, as a function of time, of being a $\tau$ neutrino?

(c) As a function of the masses and $\alpha$, what is the oscillation time? I.e. the time to return to its original flavor.

(d) If the neutrinos are extremely relativistic, $k >> m$, describe how the oscillation time translates into an oscillation as a function of the distance from the creation.

Note: Here the "masses" are the rest energies ($mc^2$).

**Solution**:
a) with $k = 0$,

$$\begin{aligned} H &= \left(\frac{m_\mu + m_\tau)}{2}\right) + \left(\frac{m_\mu - m_\tau}{2}\right)\sigma_z + \alpha\sigma_x \\ &= \left(\frac{m_\mu + m_\tau}{2}\right) + \hbar\omega\sigma_n, \end{aligned}$$

where

$$\begin{aligned} \hbar\omega &= \sqrt{(m_\mu - m_\tau)^2/4 + \alpha^2}, \\ \sigma_n &= \left[\left(\frac{m_\mu - m_\tau}{2}\right)\sigma_z + \alpha\sigma_x\right]/(\hbar\omega) \end{aligned}$$

$$\begin{aligned} U &= e^{-iHt/\hbar} \\ &= e^{-i(m_\mu + m_\tau)t/\hbar}\{\cos \omega t - i\sigma_n \sin \omega t\}. \end{aligned}$$

b)

$$P_{\mu\to\tau} = |\langle\tau|U|\mu\rangle|^2 = \frac{\alpha^2}{\hbar^2\omega^2}\sin^2\omega t.$$

c)

$$\tau_0 = \frac{\pi}{\omega}.$$

d)

$$\tau = \gamma\tau_0, \quad \gamma \approx \frac{\hbar k c}{m}.$$

18. Show that

$$\mathrm{Tr} A_S B_S C_S = \mathrm{Tr} A_H(t) B_H(t) C_H(t),$$

where the subscripts refer to Schrödinger and Heisenberg representations.

**Solution**:

$$\begin{aligned}
\mathrm{Tr} A_S B_S C_S &= \mathrm{Tr} A_S e^{iHt/\hbar} e^{-iHt/\hbar} B_S e^{iHt/\hbar} e^{-iHt/\hbar} C_S e^{iHt/\hbar} e^{-iHt/\hbar} \\
&= \mathrm{Tr} e^{-iHt/\hbar} A_S e^{iHt/\hbar} e^{-iHt/\hbar} B_S e^{iHt/\hbar} e^{-iHt/\hbar} C_S e^{iHt/\hbar} \\
&= \mathrm{Tr} A_H B_H C_H.
\end{aligned}$$