SUBJECT EXAM SAMPLE QUESTIONS

PHYSICS 851/852

1. (a) Define a projection operator P_i that projects a state i in terms of bras and kets.
   (b) Derive the relation $P^2 = P$.
   (c) Consider the 3-state basis,

$$|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \ , \ \ |\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \ , \ \ |\psi_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

   Find the projection operator that projects the state, $|+\rangle = (|\psi_1\rangle + |\psi_2\rangle)/\sqrt{2}$.
   (d) Show that $P_+^2 = P_+$.

2. Consider the $\ell = 1$ basis where $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, represent the m =1,0,-1 projections respectively of L_z. Write down the matrix components of the operator that represents a rotation by an angle ϕ about the z axis.

3. Take for granted the expansion for a plane wave,

$$e^{i\mathbf{k}\cdot\mathbf{r}} = \frac{1}{2}\sum_{\ell=0}^{\infty} i^\ell (2\ell+1) P_\ell(\cos\theta) \left\{ h_\ell(kr) + h_\ell^*(kr) \right\} \ ,$$

   where

$$h_\ell(x)|_{x\to\infty} = \frac{-i}{x} e^{ix-\ell\pi/2} \ ,$$

   and the definition of the scattering amplitude f,

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2, \psi_{\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} + f(\Omega)\frac{e^{ikr}}{r}.$$

   Derive the relation, for a spherically symmetric potential,

$$\sigma = \frac{4\pi}{k^2}\sum_\ell (2\ell+1)\sin^2\delta_\ell$$

4. Consider a particle of mass m that feels the spherically symmetric potential,

$$V(r) = \beta\delta(r-a) \ , \ \ \beta > 0$$

   (a) Find the phase shift as a function of energy.

(b) What is the scattering length?

(c) If $\beta < 0$, find the minimum value of a that permits a bound state?

5. (a) Starting from the expression for the evolution operator,

$$U(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt' V(t') U(t',t_0) \ ,$$

Derive Fermi's golden rule for the transition rate from state i to state f where the interaction is independent of time.

$$\Gamma_{f\to i} = \frac{2\pi}{\hbar}|\langle f|V|i\rangle|^2 \delta(\epsilon_f - \epsilon_i).$$

(b) If the interaction has an explicit time dependence, $V_t = V\cos\omega t$, the relevant expression becomes:

$$\Gamma_{f\to i} = \frac{2\pi}{4\hbar}|\langle f|V|i\rangle|^2 \left\{\delta(\epsilon_f - \epsilon_i - \hbar\omega) + \delta(\epsilon_f - \epsilon_i + \hbar\omega)\right\}$$

Consider a particle of mass m in the ground state of a hydrogen atom with a ground state wave function,

$$\psi_0(\mathbf{r}) = \frac{2}{\sqrt{4\pi a_0^3}} e^{-r/a_0},$$

where the binding energy is $e^2/(2a_0)$.
A perturbative force is added,

$$V_t = F\cos(\omega t).$$

Find the rate at which the atom is ionized. Your answer may contain a_0.

6. Particles of type a and b with identical masses m are mixed by the interaction,

$$V_{ab} = g\int d^3r \Psi_a^\dagger(\mathbf{r})\Psi_b(\mathbf{r}) + \text{h.c.}$$

Particle a feels an interaction with a harmonic oscillator potential,

$$V = \frac{1}{2}m\omega^2 r^2 \ ,$$

while particle b is immune to it's effects.

(a) If particle a is in the ground state of the potential, estimate the rate at which a decays into b. Use Fermi's golden rule.

(b) A particle of type b scatters off the well with an energy close to $3\hbar\omega/2$. Find an expression for the cross section which is valid for energies near $3\hbar\omega/2$. Assume $g << \hbar\omega$.

7. (a) Derive the expression for $\Delta E^{(2)}$ in second-order stationary-state perturbation theory.

(b) Assuming a particle of mass m is in a one-dimensional harmonic oscillator characterized by freqency ω, find the correction to the ground-state energy due to the interaction, $V = Fx$, in second order perturbation theory.

(c) Find the exact expression and compare to the result above.

8. For the Hamiltonian,

$$H = \sum_k \epsilon_k a_k^\dagger a_k$$

Show that in the Heisenberg representation,

(a) $a_k(t) = e^{-i\epsilon_k t} a_k$.

(b) $a_k^\dagger(t) = e^{i\epsilon_k t} a_k^\dagger$.

(c) $e^{-iHt} a_i b_j^\dagger c_k^\dagger e^{iHt} = e^{i(\epsilon_j + \epsilon_k - \epsilon_i)t} a_i b_j^\dagger c_k^\dagger$.

9. A particle of mass m is initially in the ground state of a one-dimensional infinite square well of width L, $0 < x < L$. At $t = 0$ the size is suddenly doubled such that $0 < x < 2L$.

(a) What is the probability of being in the ground state of the new well?

(b) What is the expectation of the energy, $\langle\Psi|H|\Psi\rangle$, before and after the expansion of the well?

(c) Repeat a, but assuming that the well has expanded gradually.

(d) Repead b, but again with the assumption of slow expansion.

10. A particle of charge e and mass m is in a magnetic field $\vec{B} = B\hat{z}$ and an electric field, $\vec{E} = E\hat{y}$.

(a) Using a gauge where both the electric and magnetic fields derive from a vector potential pointed along the y axis, write the Hamiltonian in Harmonic oscillator form.

(b) If the electric field is zero, what are the eigen-energies?

(c) In a two-dimensional system ($p_z = 0$), how many electrons may occupy the lowest energy level if the dimensions are L_x and L_y? (Again, assume there is no electric field.)

11. (a) Starting with the expressions,

$$\begin{aligned} \langle \ell, m|\vec{L}^2|\ell, m\rangle &= \ell(\ell+1)\hbar^2 \\ \langle \ell, m|L_z|\ell, m\rangle &= m\hbar\ , \end{aligned}$$

derive the expression

$$L_+|\ell, m\rangle \equiv (L_x + iL_y)|\ell, m\rangle = \sqrt{(\ell - m)(\ell + m + 1)}|\ell, m+1\rangle$$

(b) In terms of ℓ, m_1 and m_2, find an expression for $\langle \ell, m_1|L_x|\ell, m_2\rangle$.

12. (a) Define a density matrix ρ_Ψ for the state Ψ in terms of $|\Psi\rangle$ and $\langle\Psi|$.

(b) Show that $\langle\Psi|K|\Psi\rangle = \mathrm{Tr}\ \rho_\Psi K$.

(c) Write ρ_+ and ρ_- in matrix form for the 3-component states,

$$|+\rangle = \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \\ 0 \end{pmatrix}, \quad |-\rangle = \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \\ 0 \end{pmatrix}.$$

(d) Write a density matrix to represent an incoherent superpostion of 50% $|+\rangle$ and 50% $|-\rangle$.

13. An electron is in a spin $\uparrow$ (along z-axis) at $t = 0$. A magnetic field is applied along the x-axis. Choose the basis,

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(a) Find the evolution of the spinor, $\chi(t)$.

(b) Repeat for the case where $\vec{B} = (B/\sqrt{2})(\hat{x} + \hat{z})$.

14. A proton and neutron with magnetic moments μ_p and μ_n sit in s-wave states of a nuclear well. They exhibit a spin-spin attraction,

$$V_{s.o.} = \alpha \mathbf{S}_1 \cdot \mathbf{S}_2$$

They are also placed in a constant magnetic field, B. Find the splitting of the four degenerate states due to spin-orbit and magnetic interactions.

15. The matrix element for radiative capture for a photon of momentum $\mathbf{k}$ is of the form,

$$\mathcal{M} = \beta\vec{\epsilon} \cdot \int d^3r \phi_f^*(\mathbf{r}) \frac{i\hbar\vec{\nabla}}{m} \phi_i(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}}.$$

(a) Derive the corresponding expression for $\mathcal{M}$ in the dipole approximation.

(b) If ϕ_i refers to the $1s$ state of Hydrogen, while ϕ_f refers to the $2s$ state, which of the following matrix elements are zero?

i. $\langle f|\vec{\epsilon} \cdot \vec{r}|i\rangle$ (electric dipole)

ii. $(\vec{k} \times)\epsilon\langle f|\vec{r} \cdot \vec{p}|i\rangle$ (magnetic dipole)

iii. $\langle f|\vec{\epsilon} \cdot \vec{r}\vec{k} \cdot \vec{r}|i\rangle$ (electric quadrupole)

(c) Again considering the transition from the $2s$ to the $1s$ state, if one expands $e^{i\mathbf{k}\cdot\mathbf{r}}$ to all powers in k, what is the lowest non-zero power of k that survives the integration?

FYI:

$$\begin{aligned} \psi_{1,0}(\mathbf{r}) &= \frac{2}{\sqrt{4\pi a_0^3}} e^{-r/a_0} \\ \psi_{2,0}(\mathbf{r}) &= \frac{1}{\sqrt{4\pi(2a_0)^3}} \left(2 - \frac{r}{a_0}\right) e^{-r/(2a_0)} \end{aligned}$$

16. Consider a two component one-dimensional gas of fermions. The two species, A and B both carry spin 1/2, and have identical masses m. They are placed in an environment where the A particles feel a additional attractive potential U. Decays occur, via the emission or absorpton of photons, which change A into B and vice versa. The net density is fixed, $n = n_A + n_B$. In terms of n, m and U, find the fraction of particles of type A.

17. A ρ_0 meson $(I = 1, I_3 = 0)$ decays into two pions. Pions have $I = 1$, with the three species being π^+, π^- and π^0. What fraction of the pion pairs are π^+, π^- vs. π^0, π^0?

18. Find the possible $L - S - J$ states for 2 electrons in the 2-p shell of Carbon. Which state is the ground state?

19. The matrix element for the electromagnetic decay of an atomic d state with $m = 0$ to an p state with $m = 0$ is given by the matrix element,

$$\mathcal{M} \equiv \alpha\vec{\epsilon} \cdot \langle \ell = 2, m = 0|\vec{r}|\ell = 1, m = 0\rangle$$

where $\alpha\vec{\epsilon} \cdot \vec{R}$ is the interaction responsible for the decay, and $\vec{\epsilon}$ is the polarization vector of the outgoing photon. Assume that one has used this matrix element to calculate the decay rate for this reaction and the resulting rate is noted as Γ_{00}.

    (a) Which components of $\vec{\epsilon}$ contribute to Γ_{00}.

    (b) In terms of Γ_{00}, find the decay rates $\Gamma_{m1,m2}$ for all five d states with projection m_1 into any of the three p states with projection m_2. Assume all polarizations are included in the rate. DO NOT EVALUATE CLEBSCH-GORDAN COEFFICIENTS.

20. An electron is in a $\ell = 1$, $j = 3/2$, $M_J = 1/2$ state. Express this state as a linear combination of states in the m_ℓ, m_s basis. Show how to calculate the Clebsch-Gordan coefficients using the relations for L_+ and L_-.

21. Consider Fermi creation operators $a^\dagger$ and $b^\dagger$. Consider the Hamiltonian,

$$H = \epsilon_0(a^\dagger a + b^\dagger b) + \gamma(a^\dagger b^\dagger + ba).$$

    (a) Find operators $\alpha^\dagger$ and $\beta^\dagger$ such that the Hamiltonian can be rewritten,

    $$H = E_0 + E(\alpha^\dagger\alpha + \beta^\dagger\beta).$$

    (b) Find the constants E_0 and E in terms of ϵ_0 and γ.

    (c) If a vacuum is defined in terms of α and β,

    $$\alpha|0\rangle = 0, \quad \beta|0\rangle = 0,$$

    find the expectation $\langle 0|a^\dagger a + b^\dagger b|0\rangle$.

22. Using $e^{-x^2/(2a^2)}$ as the variational wave function with a as the variational parameter, estimate the ground state energy of a particle of mass m moving in the potential

$$V(x) = \beta|x|. \tag{1}$$

23. Consider a particle of mass m moving under the influence of a harmonic oscillator potential in two dimensions,

$$V(x,y) = \frac{1}{2}m\omega^2(x^2+y^2).$$

The eigen-energies are

$$E_n = (n+1)\hbar\omega, \quad n = 0, 1, 2, \cdots$$

(a) What is the degeneracy of the level n?

(b) If the states are eigenstates of L_z,

$$\Psi_{n,m}(x,y) \sim \psi_{n,m}(\sqrt{x^2+y^2})e^{im\phi},$$

what are the allowed values of m for a level n? Justify your answer.

(c) Now consider a modified potential,

$$V(x,y) = \frac{1}{2}m\omega^2(x^2+y^2+2\beta xy).$$

Write the Hamiltonian (including the kinetic-energy part) in terms of variables

$$R \equiv (x+y)/\sqrt{2}, \quad r \equiv (x-y)/\sqrt{2}.$$

(d) What are the eigen-energies of the new Hamiltonian?