FINAL EXAM SAMPLE QUESTIONS

PHYSICS 851, FALL 1998

1. (a) If P is the momentum operator in one-dimension and X is the position operator, write down equations of motion for the Heisenberg operators $P(t)$ and $X(t)$ given the Hamiltonian,

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2 + mC\omega^2 X$$

   (b) Consider the state prepared such that at $t = 0$,

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

   Find $\langle\psi(t)|X|\psi(t)\rangle$ and $\langle\psi(t)|P|\psi(t)\rangle$.

2. Consider a spin 1/2 system. The projection operator P_y projects the component of the wave function that has positive spin along the y axis.

$$\langle\eta|P_y|\eta\rangle = |\langle y,\uparrow|\eta\rangle|^2$$

   (a) Express P_y as a matrix in the basis where $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ denotes a state with positive spin along the z axis.

   (b) Calculate P_y^2.

3. A spin 1/2 particle has positive spin along the y axis.

   (a) Write down the density matrix ρ for this state. Again use the basis where $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ denotes a state with positive spin along the z axis.

   (b) Find ρ^2.

   (c) Write down the density matrix for the state that is an incoherent mixture of 50% positive spin along the y axis and 50% negative spin along the y axis.

   (d) What is the square of this density matrix?

4. A plane polarized photon at $\theta = 45°$ enters a special crystal with indices of refraction:
   n_x=1.50 for photons polarized along the x axis
   n_y=1.52 for photons polarized along the y axis.
   Assuming the wavelength of the light is 660 nm before it enters the crystal, choose the smallest thickness of the crystal such that the outgoing light is right circularly polarized. Assume the dispersion is linear, $k = n\omega/c$.

5. A liquid has a different index of refraction n_r for RCP photons than it's index of refraction n_l for LCP photons. Describe how such a liquid could be used to rotate linearly polarized light by 90°.

6. Consider two flavors of neutrinos, the μ neutrino and the τ neutrino. Suppose that the Hamiltonian can be written as a free term plus a term that mixes the μ and τ neutrinos, which is proportional to α.

$$\mathcal{H} = \begin{pmatrix} m_\mu c^2 & 0 \\ 0 & m_\tau c^2 \end{pmatrix} + \alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

If a neutrino starts as a μ neutrino, what is the probability, as a function of time, of being a τ neutrino?

7. A particle of mass m coming from the left interacts with a potential step of height V_0.

$$V(x) = \begin{matrix} 0, & x < 0 \\ V_0, & x > 0 \end{matrix}$$

(a) If $V_0 > 0$, what is the time delay of a reflected wave packet whose energy is narrowly centered at $E < V_0$? (The "delay" is a comparison to the time it takes to reflect from an infinite barrier.)

(b) If $V_0 < 0$, what percentage of the beam is reflected backwards.

8. Consider a particle of mass m interacting with an attractive one-dimensional potential,

$$V(x) = \begin{cases} 0, & x < 0 \\ -V_0, & 0 < x < a \\ 0, & x > a \end{cases}$$

(a) What is the ground-state energy?

(b) What is the minimum depth V_0 that permits the existence of a second bound state?

9. Consider a particle of mass m that feels an attractive one-dimensional delta function potential,

$$V(x) = -\beta\delta(x)$$

(a) Derive the ground state energy.

(b) If the well suddenly dissolves, find the differential probability of observing an asymptotic momentum state p.

10. Consider the harmonic oscillator problem, with Hamiltonian,

$$\mathcal{H} = \frac{P^2}{2m} + \frac{m\omega^2 X^2}{2}$$

(a) Calculate the following matrix elements, (in terms of ω and m)

$$\langle n|X|m\rangle \ , \ \langle n|PX|m\rangle \ , \ \langle n|P|m\rangle \ , \ \langle n|X^2|m\rangle \ ,$$

(b) Show that the expectation value of the potential energy in an energy eigenstate of the harmonic oscillator equals the expectation value of the kinetic energy in that state.

11. For a particle of mass m in the ground state of a one-dimensional harmonic oscillator in it's ground state, find the expectation

$$\langle 0|[X(t), P(t')]|0\rangle,$$

where $X(t)$ and $P(t)$ refer to the position and momentum operators in the Heisenberg representation.

12. In terms of ℓ, m_1 and m_2 find expressions for:

    (a) $\langle \ell m_1|L_x|\ell m_2\rangle$

    (b) $\langle \ell m_1|L_x^2 + L_y^2|\ell m_2\rangle$

    (c) $\langle \ell m_1|L_+L_-|\ell m_2\rangle$

13. Consider a two-dimensional rotationally symmetric harmonic oscillator with frequency ω. In a Cartesion bases the eigenenergies are $E = (n_x + n_y + 1)\hbar\omega$, where n_x and n_y are integers which are greater or equal to zero.

    (a) How many independent solutions are there for a given $N \equiv n_x + n_y$.

    (b) In terms of r and ϕ, write down the eigenstates of angular momentum that have energy $2\hbar\omega$. Express your answer as a wavefunction, $\psi(r, \phi)$.

14. A particle of mass m and charge e interacts with the vector potential

$$\begin{aligned} A_x &= 0 \\ A_y &= Bx \\ A_z &= 0 \end{aligned}$$

    (a) What is the magnetic field generated by the vector potential?

    (b) Find the ground state wave function.

    (c) Find the ground state energy.

15. What is the binding energy of an anti-proton to a Helium nucleus? (Within 1% is sufficient.) DATA: $m_p = 1836 m_e$

16. Assuming a potential of the form,

$$V(\mathbf{r}) = \beta\left[\delta^3(\mathbf{r} - a\hat{z}) + \delta^3(\mathbf{r} + a\hat{z})\right],$$

Find the differential cross section, $d\sigma/d\Omega$ of a particle of mass m as a function of the energy E, using the Born approximation.

17. Consider a particle of mass m and charge e in a harmonic oscillator with characteristic frequency ω. An electric field E is added.

    (a) In second order perturbation theory, find the correction to the ground state energy due to the electric field.

    (b) Calculate the correction to all orders, and show that it is consistent with the perturbative answer above when expanded in powers of e^2.

18. This problem is set in one dimension. An electron is in the ground state of the attractive potential,

$$V(x) = -\beta\delta(x)$$

A uniform electric field is applied which varies in time as $\mathcal{E}\cos\omega t$ Assuming that $\hbar\omega$ is greater than the binding energy of the potential, estimate the ionization rate using Fermi's golden rule.

19. A magnetic field is applied 45° to the z axis with the time dependence,

$$B(t) = \begin{cases} 0, & t < 0 \\ B_0, & 0 < t < \tau \\ 0, & t > \tau \end{cases}$$

If an electron is initially in a spin-up state (up being defined relative to the z axis), find the probability of the electron being in the spin-down state for times, $t > \tau$.

20. Consider the $\ell = 1$ basis where $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, which represent the 1,0,-1 projections respectively of L_z. Write down the matrix components of the operator that represents a rotation by an angle ϕ about the z axis.

21. Express the state $|j_1 = 1, j_2 = 2, m_1 = 1, m_2 = 1\rangle$ as a linear combination of eigenstates of total angular momentum, J and projection, M.

22. A proton and neutron with magnetic moments μ_p and μ_n sit in s-wave states of a nuclear well. They exhibit a spin-spin attraction,

$$V_{s.o.} = \alpha\mathbf{S}_1 \cdot \mathbf{S}_2$$

They are also placed in a constant magnetic field, B. Find the splitting of the four degenerate states due to spin-orbit and magnetic interactions.

23. Because of weak (neutral-current) interactions there is a parity-violating potential between the atomic electron and the nucleus as follows:

$$V = \lambda \left[\delta^3(\mathbf{x}) \mathbf{S} \cdot \mathbf{p} + \mathbf{S} \cdot \mathbf{p} \delta^3(\mathbf{x}) \right]$$

where $\mathbf{S}$ and $\mathbf{p}$ are the spin and momentum operators of the electron, and the nucleus is assumed to be situated at the origin. As a result, the ground state of an alkali atom, usually characterized by $|n, \ell, j, m\rangle$ actually conains tiny contriubtions from other eigenstates as follows

$$|n, \ell, j, m\rangle \to |n, \ell, j, m\rangle + \sum_{n', \ell', j', m'} C_{n', \ell', j', m'} |n', \ell', j', m'\rangle$$

On the basis of symmetry considerations and conservation laws, what can you say about (n', ℓ', j', m') which give rise to nonvanishing contributions?

24. Estimate the ground state energy of the Hydrogen atom, $V = -e^2/r$, using a Gaussian wave function,

$$\psi(\mathbf{r}) \propto e^{-r^2/(2a^2}.$$

Express your answer in terms of $\hbar$, e and the electron mass m.

Useful information: In a one-dimensional harmonic oscillator where the ground state wave function is proportional to $e^{-x^2/(2a^2)}$, the expectation of the kinetic energy is $\hbar^2/(4ma^2)$.