CHEAT SHEET for PHYSICS 851/852 FINALS

Equations of Motion and Evolution

$$\frac{d\mathcal{O}}{dt} = \frac{i}{\hbar}[H, \mathcal{O}]$$

$$U(t, t_0) = e^{-iH(t-t_0)/\hbar}$$

$$\psi_p(x, t) = e^{-iEt/\hbar + ipx/\hbar}$$

The Scattering amplitude f

$$\psi(\mathbf{r}, t) = \psi_0(\mathbf{r}, t) + \frac{f_{k_0}(\Omega)}{r} e^{ikr}$$

$$\frac{d\sigma}{d\Omega} = |f_{k_0}(\Omega)|^2$$

The Born approximation:

$$f(\Omega) \approx -\frac{m}{2\pi\hbar^2} \int d^3r' e^{i\mathbf{k}-\mathbf{k}'\cdot\mathbf{r}'} v(\mathbf{r}')$$

Symmetric Potentials and Partial Waves

In terms of Hankel functions,

$$\begin{aligned} e^{i\mathbf{k}\cdot\mathbf{r}} &= \frac{1}{2}\sum_{\ell=0}^{\infty} i^\ell (2\ell+1) P_\ell(\cos\theta) \left[h_\ell(kr) + h_\ell^*(kr),\right] \\ &= \sum_{\ell=0}^{\infty} i^\ell (2\ell+1) P_\ell(\cos\theta) j_\ell(kr) \end{aligned}$$

where k is along z axis.

Spherical Bessel functions,

$$j_\ell \equiv \frac{1}{2}(h_\ell + h_\ell^*)$$

Spherical Neumann functions,

$$n_\ell \equiv \frac{1}{2i}(h_\ell - h_\ell^*)$$

Some examples:

$$\begin{aligned} j_0(x) &= \sin(x)/x \\ j_1(x) &= \sin(x)/x^2 - \cos(x)/x \\ n_0(x) &= \cos(x)/x \\ n_1(x) &= -\sin(x)/x - \cos(x)/x^2 \end{aligned}$$

Expansion at small x:

$$
\begin{aligned}
j_\ell &\propto x^\ell \\
n_\ell &\propto 1/x^{\ell+1}
\end{aligned}
$$

Asymptotic behavior at large x:

$$h_\ell(x) \approx i^{-(\ell+1)} \frac{e^{ix}}{x}$$

Effect of potential $V(r)$:

$$\begin{aligned}
h_\ell &\rightarrow e^{2i\delta} h_\ell \\
\psi_{\mathbf{k}}(\mathbf{r}) &= \frac{1}{2}\sum_{\ell=0}^{\infty} i^\ell (2\ell+1) P_\ell(\cos\theta)\left[h_\ell(kr)e^{2i\delta} + h_\ell^*(kr)\right] \\
&= e^{i\mathbf{k}\cdot\mathbf{r}} + \sum_\ell i^{\ell+1}(2\ell+1)P_\ell(\cos\theta)e^{i\delta_\ell}\sin(\delta_\ell)h_\ell(kr) \\
f(\theta) &= \frac{1}{k}\sum_\ell (2\ell+1)P_\ell(\cos\theta)e^{i\delta_\ell}\sin(\delta_\ell)
\end{aligned}$$

Angle-integrated cross section:

$$\begin{aligned}
\sigma &= \sum_\ell \sigma_\ell \\
\sigma_\ell &= \frac{4\pi}{k^2}(2\ell+1)\sin^2\delta_\ell
\end{aligned}$$

Solving for wave functions and phase shifts with a potential V

$$\text{If } u_\ell(kr) \equiv krR_\ell(kr),$$

where $R_\ell(kr)Y_{lm}(\theta\phi)$ is solution of 3-d Schroedinger Eq., then,

$$\left(-\frac{\hbar^2}{2m}\partial_r^2 + \frac{\ell(\ell+1)}{2mr^2} + V(r)\right)u_\ell = Eu_\ell$$

Algorithm: (1) Find u_ℓ, (2) $R_\ell = u_\ell/(kr)$, (3) Phase shifts found by matching $R_\ell \propto h_\ell^* + e^{2i\delta}h_\ell$ at large kr (large enough s.t. $V = 0$).

Born Approximation for phase shifts:

$$\delta_\ell \approx -\frac{2mk}{\hbar^2}\int r^2 dr v(r) j_\ell(kr)^2$$

Near resonance,

$$\tan\delta_\ell \approx -\frac{\Gamma}{2(E-E_r)}$$

Phase shifts at low momentum,

$$\delta_l \propto k^{(2\ell+1)}$$

Scattering length, a,

$$\cot\delta_0 \approx -\frac{1}{ka}, \quad \delta_0 \approx -ka$$

$$\sigma(k=0) = 4\pi a^2$$

Hydrogen Atom

Bohr radius, $a_0 \equiv \hbar^2/(me^2)$

Energy levels,

$$E_n = -\frac{e^2}{2a_0}\frac{1}{n^2}.$$

Radial wave functions:

$$\begin{aligned}
R_{1,0} &= \frac{2}{a_0^{3/2}} e^{-r/a_0} \\
R_{2,0} &= \frac{1}{(2a_0)^{3/2}}\left(2 - \frac{r}{a_0}\right) e^{-r/(2a_0)} \\
R_{2,1} &= \frac{1}{(2a_0)^{3/2}} \frac{r}{a_0\sqrt{3}} e^{-r/(2a_0)}
\end{aligned}$$

Coulomb Scattering

$$\psi_{sc}(r) = \frac{e^{ikr - i\eta\log(2kr)}}{r} f_c(\theta)$$

where $\eta \equiv Z_1 Z_2 e^2/(\hbar^2 k)$ is the Sommerfeld parameter.

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{16E^2}\frac{1}{\sin^4(\theta/2)}$$

Gamow penetration factor,

$$|\psi(r=0)|^2 = \frac{2\pi\eta}{e^{2\pi\eta} - 1}$$

Fermi's Golden rule

$$\Gamma = \frac{2\pi}{\hbar}|V_{mn}|^2 \delta(E_m - E_n)$$

For harmonic potentials, $V_t = V\cos(\omega t)$,

$$\Gamma = \frac{2\pi}{\hbar}\frac{1}{4}|V_{mn}|^2 \left\{\delta(E_m - E_n - \hbar\omega) + \delta(E_m - E_n + \hbar\omega)\right\}$$

Angular Momentum Raising and Lowering

$$\begin{aligned}
L_+|\ell, m\rangle &= \sqrt{\ell(\ell+1) - m(m+1)}|\ell, m+1\rangle \\
L_-|\ell, m\rangle &= \sqrt{\ell(\ell+1) - m(m-1)}|\ell, m-1\rangle
\end{aligned}$$

For a particle of charge e in a magnetic field,

$$\mathcal{H} = -\frac{e}{2mc}\vec{B}\cdot\left\{\vec{\ell}+g\vec{s}\right\},$$

where $g = 2$ for an electron.

Harmonic Oscillators
G.S. wave function of 1-d harmonic oscillator:

$$\psi(x) = \frac{1}{a^{1/2}\pi^{1/4}}\exp\frac{-x^2}{2a^2},$$

where $a^2 = \hbar/(m\omega)$.

Creation and destruction operators:

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}X - i\sqrt{\frac{1}{2m\hbar\omega}}P$$

$$X = \sqrt{\frac{\hbar}{2m\omega}}\left(a^\dagger + a\right), \quad P = i\sqrt{\frac{\hbar m\omega}{2}}\left(a^\dagger - a\right)$$

Interaction with Electromagnetic field

$$H = \frac{(\vec{p}-\frac{e}{c}\vec{A})^2}{2m} + e\Phi$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

$$\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$$

Density of states In n dimensions,

$$dN = \Omega^{(n)}\frac{d^n p}{(2\pi\hbar)^n},$$

where $\Omega^{(n)}$ is the length, area, volume $\cdots$

Resonant Scattering For particles of types 1 and 2 scattering through a resonance,

$$\sigma = \frac{(2S_R+1)}{(2S_1+1)(2S_2+1)}\frac{4\pi}{k^2}\frac{(\hbar\Gamma/2)^2}{(E-E_R)^2+(\hbar\Gamma/2)^2}$$

Spherical Harmonics

$$
\begin{array}{ll}
\ell = 0 & Y_{00} = \frac{1}{\sqrt{4\pi}} \\
\ell = 1 & \left\{ \begin{array}{l} Y_{1\pm1} = \mp\sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi} \\ Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta \end{array} \right. \\
\ell = 2 & \left\{ \begin{array}{l} Y_{2\pm2} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta e^{\pm 2i\phi} \\ Y_{2\pm1} = \mp\sqrt{\frac{15}{8\pi}} \sin\theta\cos\theta e^{\pm i\phi} \\ Y_{20} = \frac{1}{2}\sqrt{\frac{5}{4\pi}}(3\cos^2\theta - 1) \end{array} \right.
\end{array}
\tag{1}
$$

Sigma Matrices

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Gamma Functions and Related Integrals

$$\Gamma(n) \equiv \int_0^\infty dx\ x^{n-1} e^{-x} = (n-1)!,$$

$$\Gamma(1/2) = \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} dx e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}$$

$$\int_{-\infty}^{\infty} dx e^{-x^2/(2a^2)} = a\sqrt{2\pi}$$

$$\int_{-\infty}^{\infty} x^{2n} e^{-\alpha x^2} dx = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n\alpha^n}\sqrt{\frac{\pi}{\alpha}}$$

Wigner-Eckart Theorem

$$\langle j_2, m_2 | T_q^k | j_1, m_1 \rangle = \langle j_2, m_2 | k, q, j_1, m_1 \rangle \frac{\langle j_2 || T^{(k)} || j_1 \rangle}{\sqrt{2j_2+1}}$$