your name(s)____

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Physics 852 Quiz #10 - Friday, March. 26th

Coulomb's Law

In Chapter 9 the electromagnetic field operator was written as,

$$\vec{A}(\vec{r},t) = \sqrt{\frac{2\pi\hbar^2 c^2}{V}} \sum_{\vec{k},s} \frac{1}{\sqrt{E(\vec{k})}} \left(\vec{\epsilon}_s(\vec{k}) e^{i\vec{k}\cdot\vec{r} - iE(\vec{k})t/\hbar} a_{\vec{k},s} + \vec{\epsilon}_s^{\,*}(\vec{k}) e^{-i\vec{k}\cdot\vec{r} + iE(\vec{k})t/\hbar} a^\dagger_{\vec{k},s} \right).$$

This ignores the zero$^{\rm th}$ component of A because there are only two polarizations, and neither has a zero$^{\rm th}$ component. Although there are no physical, on-shell, phonons with polarizations either parallel to $\vec{k}$, or in the zero$^{\rm th}$ direction, one can write the field with all four components μ,

$$A_\mu(\vec{r},t) = \sqrt{\frac{2\pi\hbar^2 c^2}{V}} \sum_{\vec{k},s} \frac{1}{\sqrt{E(\vec{k})}} \left(\epsilon_{s,\mu}(\vec{k}) e^{ik\cdot x} a_{\vec{k},s} + \epsilon^*_{s,\mu}(\vec{k}) e^{-ik\cdot x} a^\dagger_{\vec{k},s} \right).$$

Next, we consider four polarization vectors (even though only two are physical) as $\epsilon_{\mu=0} = (1,0,0,0)$, $\epsilon_{\mu=1} = (0,1,0,0), \cdots$. There is no longer any need to write down polarization vectors, because the creation/destruction operators now behave as four vectors. Note that x refers to the four-vector $t, \vec{r}$ and $k \cdot x = Et/\hbar - \vec{k}\cdot\vec{r}$. The field is simply,

$$A_\mu(\vec{r},t) = \sqrt{\frac{2\pi\hbar^2 c^2}{V}} \sum_{\vec{k}} \frac{1}{\sqrt{E(\vec{k})}} \left(e^{ik\cdot x} a_{\vec{k},\mu} + e^{-ik\cdot x} a^\dagger_{\vec{k},\mu} \right).$$

However, in order for the expression for A^μ to behave as a four vector the commutation rules are

$$[a_{\mu,\vec{k}}, a^\dagger_{\nu,\vec{k}'}] = -g_{\mu\nu}\delta_{\vec{k}\vec{k}'}.$$

For $\mu = 1,2,3$ these are the usual commutation rules because $-g_{ii} = 1$. However, the signs are reversed for the zero$^{\rm th}$ component, as if the probability of that state is negative. For a state with polarization $(1,0,0,0)$ the norm of the state is

$$\begin{aligned} \langle \vec{k}, \mu = 0 | \vec{k}, \mu = 0 \rangle &= \langle 0 | (a^\dagger_{\vec{k},\mu=0})^\dagger a^\dagger_{\vec{k},\mu=0} | 0 \rangle \\ &= \langle 0 | a_{\vec{k},\mu=0} a^\dagger_{\vec{k},\mu=0} | 0 \rangle \\ &= -1. \end{aligned}$$

Where the last step used the commutation laws. Thus, these creation and destruction operators have the peculiar behavior that the states seem to have negative norms.

Your mission, should you choose to accept it, is to consider a static charge density,

$$j_0(\vec{r}) = \rho(\vec{r}) \neq 0,\ \vec{j}(\vec{r}) = 0,$$

and derive and expression for the energy due to interaction with the quantum electromagnetic field in second order perturbation theory.

Here, the charge density is fixed (or classical), and can be considered as a given function of $\vec{r}$, i.e. it is not altered by the interaction with the electromagnetic field. The state $|0\rangle$ refers to a state with zero photons (the ground state), but it does have the static charge density $\rho(\vec{r})$. The state $|\mu, \vec{k}\rangle$ refers to a field with the same charge density but with a single photon of wave vector $\vec{k}$ and polarization μ.

The correction to the ground state energy in second order perturbation theory is

$$\begin{aligned}
\delta E^{(2)} &= -\sum_{i \neq 0} \frac{|\langle \mu, \vec{k}|H_{\text{int}}|0\rangle|^2}{E_i}, \\
H_{\text{int}} &= \int d^3r\, j^\alpha(\vec{r}) A_\alpha(\vec{r}), \\
E(\vec{k}) &= \hbar c k.
\end{aligned}$$

1. Using the definitions above, show that $\delta E^{(2)}$ can be written as

$$\delta E^{(2)} = \int d^3r_1 d^3r_2\, \frac{\hbar^2 c^2}{(2\pi)^2} \int \frac{d^3k}{E(\vec{k})^2}\, \rho(\vec{r}_1)\rho(\vec{r}_2) e^{i\vec{k}\cdot(\vec{r}_1 - \vec{r}_2)}\ .$$

2. Perform the integral over $\vec{k}$ to derive Coulomb's law,

$$\delta E^{(2)} = \frac{1}{2} \int d^3r_1 d^2r_2\, \frac{\rho(\vec{r}_1)\rho(\vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|}\ .$$

Note that the energy is positive (for any $\rho(\vec{r})$). However, from the expression for $\delta E^{(2)}$ one would would expect that the second-order correction to the ground state energy should always be negative. The source of this paradox comes from the opposite signs in the definitions of the commutation relation. One immediately asks the question of why one introduces two polarizations (known as longitudinal) that shouldn't exist, one with peculiar creation/destruction operators. The answer is that if one calculates the probability for emitting into either of these unphysical states, the probabilities come out as equal and opposite. Thus, the negative probability is always canceled by additional positive probability, so that one can safely neglect the contribution from them if BOTH are left out of any calculation where they appear in the initial or final state. However, for intermediate states, such as above, they cannot be neglected. This particular form for the field operator is known as the "Feynmann Gauge". The simple fact that the Coulomb energy is always positive drives home the point that electromagnetic theory must incorporate some sort of peculiar behavior, whether that be in the form of having $\langle 0|aa^\dagger|0\rangle = -1$, or in introducing "ghost" fields (for another course).