your name(s)____

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Physics 852 Exercise 12 - Friday, March 25

Scalars, Four-Vectors and Lorentz Boosts

1) Consider a four-velocity along the $x-$axis,

$$u^\mu = \begin{pmatrix} \gamma \\ \gamma v \\ 0 \\ 0 \end{pmatrix}, \quad \gamma \equiv \frac{1}{\sqrt{1-v^2}}.$$

Show that $u^2 = 1$.

2) The matrix that boost a four-vector by this velocity is

$$L^\mu{}_\nu = \begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By considering the Jacobian of the transformation, show that $d^4p = dp_0 dp_x dp_y dp_z$ is a Lorentz invariant.

3) Using the fact that $\delta(p^2 - m^2)$ is a scalar, where m is the mass and p is a four-vector, show that d^3p/E_p is a Lorentz invariant, where $E_p = \sqrt{|\vec{p}|^2 + m^2}$.

4) Using (3), show that $(E_p + E_{p'})\delta(\vec{p} - \vec{p}')$ is a Lorentz invariant. Hint: Consider the indefinite integral

$$\int \frac{d^3p'}{E_{p'}}(E_p + E_{p'})\delta(\vec{p} - \vec{p}').$$

FYI: This means that the field operator

$$\Phi(x) = \int \frac{d^3p}{E}\left\{a(\vec{p})e^{-ip\cdot x} + a^\dagger(\vec{p})e^{ip\cdot x}\right\}$$

is a Lorentz scalar if one defines the creation and destruction operators as $[a(\vec{p}), a^\dagger(\vec{p}')] = E_p\delta(\vec{p} - \vec{p}')$.

5) Consider two four velocities, u and n, where n represents the lab frame,

$$n^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Show that the Lorentz boost matrix defined as,

$$L^{\mu\nu}(u,n) = g^{\mu\nu} + 2u^\mu n^\nu - \frac{(u^\mu + n^\mu)(u^\nu + n^\nu)}{1 + u\cdot n},$$

reproduces L in (2) when $\vec{u}$ is along the x axis.

6) Show that

$$L^{\mu\nu}n_\nu = u^\mu.$$

7) Show that

$$L^{\mu\alpha}(n,u)L_\alpha{}^\nu(u,n) = g^{\mu\nu}.$$