your name(s)____

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Physics 852 Exercise #11 - Friday, April. 9th

Consider the scalar field operator (in one dimension),

$$\Phi(x,t) = \frac{1}{\sqrt{L}} \sum_k \frac{1}{\sqrt{2\omega_k}} \left[a_k e^{-i\omega_k t + ikx} + a_k^\dagger e^{i\omega_k t - ikx} \right],$$

$\omega_k = E_k/\hbar$. One creates a state $|\eta\rangle$, which for positive times is defined as

$$|\eta(t)\rangle_I = \exp\left\{ i \int dt' dx \, j(x,t')\Phi(x,t') \right\} |0\rangle,$$

where $j(x,t')$ is a real function and is zero for $t' > t$.

1. Find the commutation relation for $[\Phi(x,t), \dot{\Phi}(x',t)]$.
2. Show that the $\langle\eta|\eta\rangle = 1$. (hint: should be very short)
3. Rewrite $|\eta(t)\rangle$ using the operators a_k and $a_k^\dagger$ instead of $\Phi(x,t)$, and assume that $j(t' > t) = 0$. Your expression should use the Fourier transforms of $j(x,t)$,
$$\tilde{j}(k,\omega) \equiv \int dx \, dt \, e^{-i\omega t + ikx} j(x,t).$$
4. Find an expression for $\langle\eta|a_k^\dagger a_k|\eta\rangle$. Note that $\langle\eta|a_k^\dagger a_k|\eta\rangle$ in the Schrödinger representation is the same as $\langle\eta(t)|a_k^\dagger(t) a_k(t)|\eta(t)\rangle$ in the Heisenberg representation.
5. Find an expression for the net number of particles.
6. Find an expression for $\langle\eta|a_k^\dagger a_q^\dagger a_q a_k|\eta\rangle$.

1.)

$$
\begin{aligned}
[\Phi(x,t),\dot{\Phi}(x',t)] &= \sum_{kq}\frac{1}{2L\sqrt{\omega_k\omega_q}}e^{-i(\omega_k-\omega_q)t+ikx-iqx'}\left\{i\omega_q[a_k,a_q^\dagger]-i\omega_q[a_k^\dagger,a_q]\right\}\\
&= i\frac{1}{L}\sum_k e^{ik(x-x')}\\
&= i\frac{1}{2\pi}\int dk\, e^{ik(x-x')}\\
&= i\delta(x-x').
\end{aligned}
$$

2.)Φ and j are Hermitian, thus $A=\int dt'dxj(x,t')\Phi(x,t')$ is Hermitian and $e^{-iAt}e^{iAt}=1$.

3.)

$$
\begin{aligned}
|\eta(t)\rangle_I - \exp&\left\{i\int_{-\infty}^{t}dt'dx\,j(x,t')\sum_k\frac{1}{\sqrt{2\omega_k L}}(a_k e^{-i\omega t+ikx}+a_k^\dagger e^{i\omega t-ikx})\right\}|0\rangle\\
&= \exp\left\{i\sum_k\frac{1}{\sqrt{2\omega_k L}}\tilde{j}(k,\omega_k)a_k + i\sum_k\frac{1}{\sqrt{2\omega_k L}}\tilde{j}(-k,-\omega_k)a_k^\dagger\right\}|0\rangle.
\end{aligned}
$$

$$\tilde{j}(-k,-\omega)=\tilde{j}^*(k,\omega).$$

4.

$$
\begin{aligned}
\langle\eta(t)|a_k^\dagger(t)a_k(t)|\eta(t)\rangle &= \langle\eta(t)|a_k^\dagger a_k|\eta(t)\rangle\\
&= \frac{1}{2\omega_k L}\left|\tilde{j}(k,\omega_k)\right|^2
\end{aligned}
$$

5.

$$
\begin{aligned}
\langle N\rangle &= \sum_k\langle\eta(t)|a_k^\dagger a_k|\eta(t)\rangle\\
&= \sum_k\frac{1}{2\omega_k L}\left|\tilde{j}(k,\omega_k)|^2\right|^2\\
&= \frac{1}{2\pi}\int\frac{dk}{2\omega_k}\left|\tilde{j}(k,\omega_k)|^2\right|^2.
\end{aligned}
$$

6.

$$
\begin{aligned}
&\langle\eta(t)|a_k^\dagger a_q^\dagger a_q a_k|\eta(t)\rangle\\
&\qquad= \frac{1}{4\omega_k\omega_q L^2}\left|\tilde{j}(k,\omega_k)|^2\right|^2\left|\tilde{j}(q,\omega_q)|^2\right|^2
\end{aligned}
$$