your name(s)____

____

Physics 851 Exercise #3

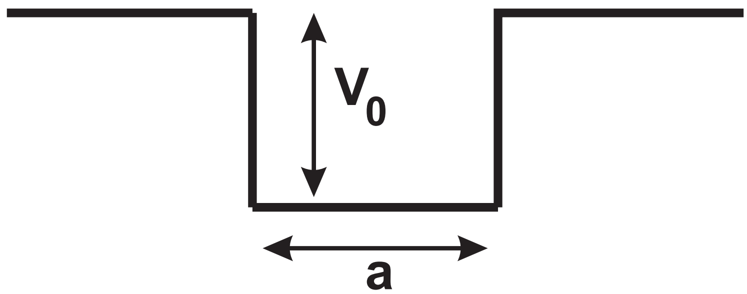
A beam of particles of N particles of mass m and momentum p has a wave function spread over a large length L,

$$\psi_p(x) = \frac{e^{ipx/\hbar}}{\sqrt{L}}.$$

While the beam is passing by the origin, a potential suddenly appears,

$$V(x) = \begin{cases} 0, & t < 0 \\ -V_0\Theta(x + a/2)\Theta(a/2 - x), & t > 0 \end{cases},$$

where Θ is a step function.



The depth of the potential is adjusted so that the ground state energy is $-V_0/2$.

Assume the normalized wave function of the ground state has the form,

$$\phi(x) = Z^{-1/2} \begin{cases} \cos(kx), & |x| < a/2, \\ Ae^{-q(|x|-a/2)}, & |x| > a/2 \end{cases}.$$

1. What are q and k in terms of V_0 and m?

   Solution:
   The energy is $E = -V_0/2 = -\hbar^2 q^2/2m$ is the same as the kinetic energy inside the well, $\hbar^2 k^2/2$, aside from the sign.

   $$q = k = \sqrt{\frac{2m(V_0/2)}{\hbar^2}}$$

2. What is A in terms of q?

   Solution:
   The first BC is

   $$\cos(ka/2) = A.$$

3. What is Z in terms of q and a?

Solution:
You need to do some integrals,

$$Z = 2A\int_{a/2}^{\infty} dx\; e^{-2q(x-a/2)} + \int_{-a/2} a/2dx\;\; \cos^2(kx) \tag{0.1}$$
$$= \frac{A}{q} + \frac{1}{2}\int_{-a/2} a/2dx\; (1+\cos(2kx))$$
$$= \frac{A}{q} + \frac{a}{2} + \frac{\sin(ka)}{2k}.$$

4. If there is a single particle, what is the probability it will fall into the ground state? Express your answer in terms of q, a, A and Z.

Solution:
Square the overlap between the plane wave and the ground state.

$$P_0 = \frac{1}{L}\left|\alpha\right|^2,$$
$$\alpha \equiv \int_{-\infty}^{\infty} dx\; e^{-ipx/\hbar}\psi_0(x)$$
$$= \alpha_I + \alpha_{II} + \alpha_{III},$$
$$\alpha_{III} = \int_{a/2}^{\infty} dx\; e^{ipx/\hbar} A e^{-q(x-a/2)}$$
$$= Ae^{ipa/2\hbar}\int_{a/2}^{\infty} dx\; e^{ip(x-a/2)/\hbar} e^{-q(x-a/2)}$$
$$= Ae^{ipa/2\hbar}\frac{1}{q-ip/\hbar},$$
$$\alpha_I = e^{-ipa/2\hbar}\int_{-\infty}^{a/2} dx\; e^{ip(x+a/2)/\hbar} A e^{q(x+a/2)}$$
$$= Ae^{-ipa/2\hbar}\frac{1}{q+ip/\hbar},$$
$$\alpha_I + \alpha_{II} = 2A\cos(pa/2\hbar)\frac{q}{q^2+p^2/\hbar^2} - 2A\sin(pa/2\hbar)\frac{p/\hbar}{q^2+p^2/\hbar^2},$$
$$\alpha_{II} = \int_{-a/2}^{a/2} dx\;\; \cos(px/\hbar)\cos(kx)$$
$$= \frac{1}{2}\int_{-a/2}^{a/2} dx\;\; [\cos(px/\hbar + kx) + \cos(px/\hbar - kx)]$$
$$= \frac{\sin(pa/2\hbar + ka/2)}{p\hbar + k} + \frac{\sin(pa/2\hbar - ka/2)}{p/\hbar - k}.$$

5. In terms of q, a, A, Z and the density (number per unit length), $\rho = N/L$, what is the average number of particles that will be in the ground state at large times?

Solution:

$$\bar{N} = P_0 N = \frac{\alpha^2}{L}\rho L = \rho\alpha^2.$$

6. For $t < 0$ how many states of momentum p are there per differential momentum, i.e. what is dN_{states}/dp?

Solution:
The BC for an infinite well of length L gives

$$\begin{aligned} pL/\hbar &= N\pi, \quad N = 1, 2, 3, \cdots \\ \frac{dN}{dp} &= \frac{L}{\pi\hbar}, \text{ (eigenstates of positive)}p \\ \frac{dN}{dp} &= \frac{L}{2\pi\hbar}, \text{ (momentum states)}, -\infty < p < \infty \end{aligned} \tag{0.2}$$

7. For $t < 0$ what is the average occupancy of a momentum state with momentum p if the momentum distribution is proportional to $e^{-E/T}$?

Solution:
Normalize the Gaussian, i.e. find f_0

$$\begin{aligned} N &= \sum_p f_0 e^{-E/T} \\ &= \frac{L}{2\pi\hbar}\int dp\ f_0 e^{-p^2/2mT}, \\ &= f_0 \frac{L}{2\pi\hbar}\sqrt{2\pi m T}, \\ f_0 &= \rho\sqrt{\frac{2\pi\hbar^2}{mT}}, \\ f(p) &= f_0 e^{-p^2/2mT}. \end{aligned}$$

8. Assuming the thermal distribution above, write an integral to express out how many particles are in the ground state for $t > 0$. (Don't perform the integral)

Solution:

$$\begin{aligned} \bar{N}_{\text{tot}} &= \sum_p f(p) P_0(p) \\ &= L\int \frac{dp}{2\pi\hbar} f(p)\frac{\alpha^2(p)}{L} \\ &= f_0 \int \frac{dp}{2\pi\hbar} e^{-p^2/2mT}\alpha^2(p). \end{aligned}$$