FINAL (SUBJECT) EXAM,
PHYSICS 852, Spring 2020
May 29-30, 3:00-3:00 PM

SECRET STUDENT NUMBER:
STUDNUMBER

1. Images of answers must be sent in within 5 minutes of finishing time by email to mailto:prattsc@msu.edu. The easiest way to submit answers is probably taking a picture by phone and sending photos in jpeg format or pdf format. Please use resolution of 300 dpi or file sizes of $\approx$ 500-600 kB per page. If you can make the images grayscale, that will help with file size. Solutions should be written on separate pages from the exam or the equation sheet, so as not to waste space, and submit only your solutions. Solutions for different problems should appear on separate pages. The MSU email system has a file size limit of 25 MB for one message. If possible, send all the images as part of a single email message. You should probably make sure your upload is using wifi, as cellular might be slow and unreliable.
2. **Do not write your name on the exam. Your exam has a "secret student number" on each page. Write that number on EACH page of the solutions you submit**.
3. This exam is open-book, open-notes and open-internet. You are permitted to use mathematical software, e.g. mathematica. However, you are not to communicate with any other individuals regarding the exam, either during the exam, or during the following 12 hours.
4. This exam has four problems valued at 100 net points. Compared to the usual format, grading will be stricter. A litany of superfluous expressions will be interpreted as evidence of a student's lack of understanding. Thus, you should not include your scratch work, only your solutions, along with clear explanations of your reasoning. If you perform math with some program, such as Mathematica, just mention how the result of the expression was obtained with said software. Problems will be more unique as to diminish the effectiveness of copy and paste from available solutions. The problems will be only slightly longer to write up, but will typically require more thinking than a usual exam as the questions are not as "standard" as in the 3-hour format.

$$\int_{-\infty}^{\infty} dx\ e^{-x^2/(2a^2)} = a\sqrt{2\pi},$$

$$H = i\hbar\partial_t,\ \vec{P} = -i\hbar\nabla,$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$U(t,-\infty) = 1 + \frac{-i}{\hbar}\int_{-\infty}^{t} dt'\ V(t')U(t',-\infty),$$

$$\langle x|x'\rangle = \delta(x-x'),\ \langle p|p'\rangle = \frac{1}{2\pi\hbar}\delta(p-p'),$$

$$|p\rangle = \int dx\ |x\rangle e^{ipx/\hbar}, \quad |x\rangle = \int \frac{dp}{2\pi\hbar}|p\rangle e^{-ipx/\hbar},$$

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \hbar\omega(a^\dagger a + 1/2),$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}X - i\sqrt{\frac{1}{2\hbar m\omega}}P,$$

$$\psi_0(x) = \frac{1}{(\pi b^2)^{1/4}}e^{-x^2/2b^2}, \quad b^2 = \frac{\hbar}{m\omega},$$

$$\rho(\vec{r},t) = \psi^*(\vec{r}_1,t_1)\psi(\vec{r}_2,t_2)$$

$$\begin{aligned} \vec{j}(\vec{r},t) &= \frac{-i\hbar}{2m}(\psi^*(\vec{r},t)\nabla\psi(\vec{r},t) - (\nabla\psi^*(\vec{r},t))\psi(\vec{r},t)) \\ &- \frac{e\vec{A}}{mc}|\psi(\vec{r},t)|^2. \end{aligned}$$

$$H = \frac{(\vec{P} - e\vec{A}/c)^2}{2m} + e\Phi,$$

For $V = \beta\delta(x-y)$: $$-\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\psi(x)|_{y+\epsilon} - \frac{\partial}{\partial x}\psi(x)|_{y-\epsilon}\right) = -\beta\psi(y),$$

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\partial_t\vec{A}, \quad \vec{B} = \nabla\times\vec{A},$$

$$\omega_{\text{cyclotron}} = \frac{eB}{mc},$$

$$e^{A+B} = e^A e^B e^{-C/2}, \quad \text{if } [A,B] = C, \text{ and } [C,A] = [C,B] = 0,$$

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}, \quad Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta, \quad Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\pm\phi},$$

$$Y_{2,0} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1), \quad Y_{2,\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi},$$

$$Y_{2,\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\phi}, \quad Y_{\ell -m}(\theta,\phi) = (-1)^m Y^*_{\ell m}(\theta,\phi).$$

$$|N\rangle = |n\rangle - \sum_{m\neq n} |m\rangle \frac{1}{\epsilon_m - \epsilon_n} \langle m|V|n\rangle + \cdots$$

$$E_N = \epsilon_n + \langle n|V|n\rangle - \sum_{m\neq n} \frac{|\langle m|V|n\rangle|^2}{\epsilon_m - \epsilon_n}$$

$$j_0(x) = \frac{\sin x}{x},\ n_0(x) = -\frac{\cos x}{x},\ j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x},\ n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x,\ n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x,$$

$$\frac{d}{dt}P_{i\to n}(t) = \frac{2\pi}{\hbar}|V_{ni}|^2\delta(E_n - E_i),$$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}\left|\int d^3r\mathcal{V}(r)e^{i(\vec{k}_f - \vec{k}_i)\cdot\vec{r}}\right|^2,$$

$$\sigma = \frac{(2S_R+1)}{(2S_1+1)(2S_2+1)}\frac{4\pi}{k^2}\frac{(\hbar\Gamma_R/2)^2}{(\epsilon_k - \epsilon_r)^2 + (\hbar\Gamma_R/2)^2},$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{single}}\tilde{S}(\vec{q}),\quad \tilde{S}(\vec{q}) = \left|\sum_{\delta\vec{a}} e^{i\vec{q}\cdot\delta\vec{a}}\right|^2,$$

$$e^{i\vec{k}\cdot\vec{r}} = \sum_{\ell}(2\ell+1)i^{\ell}j_{\ell}(kr)P_{\ell}(\cos\theta),$$

$$P_{\ell}(\cos\theta) = \sqrt{\frac{4\pi}{2\ell+1}}Y_{\ell,m=0}(\theta,\phi),$$

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = (3x^2-1)/3,$$

$$f(\Omega) \equiv \sum_{\ell}(2\ell+1)e^{i\delta_\ell}\sin\delta_\ell\frac{1}{k}P_{\ell}(\cos\theta)$$

$$\psi_{\vec{k}}(\vec{r})|_{R\to\infty} = e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r}f(\Omega),$$

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2,\quad \sigma = \frac{4\pi}{k^2}\sum_{\ell}(2\ell+1)\sin^2\delta_\ell,$$

$$L_{\pm}|\ell,m\rangle = \sqrt{\ell(\ell+1) - m(m\pm1)}|\ell,m\pm1\rangle,$$

$$C^{\ell,s}_{m_\ell,m_s;JM} = \langle\ell,s,J,M|\ell,s,m_\ell,m_s\rangle,$$

$$\langle\tilde{\beta},J,M|T^k_q|\beta,\ell,m_\ell\rangle = C^{k\ell}_{qm_\ell;JM}\frac{\langle\tilde{\beta},J||T^{(k)}||\beta,\ell,J\rangle}{\sqrt{2J+1}},$$

$$n = \frac{(2s+1)}{(2\pi)^d}\int_{k<k_f} d^dk,\quad d\text{ dimensions},$$

$$\{\Psi_s(\vec{x}),\Psi^\dagger_{s'}(\vec{y})\} = \delta^3(\vec{x}-\vec{y})\delta_{ss'},$$

$$\Psi^\dagger_s(\vec{r}) = \frac{1}{\sqrt{V}}\sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}a^\dagger_s(\vec{k}),\quad \{\Psi_s(\vec{x}),a^\dagger_\alpha\} = \phi_{\alpha,s}(\vec{x}).$$

1. Consider the states

$$|\alpha\rangle = \exp(\alpha a^\dagger + \alpha^* a)|0\rangle, \quad |\beta\rangle = \exp(\beta a^\dagger - \beta^* a)|0\rangle,$$

where α and β are complex numbers and $a^\dagger$ and a are Bosonic creation and destruction operators.

(a) (10 pts) What are $\langle\alpha|\alpha\rangle$ and $\langle\beta|\beta\rangle$?

(b) (10 pts) Consider a time-dependent Hamiltonian

$$H(t) = E_0 a^\dagger a + J_0 \Theta(t)(a + a^\dagger),$$

where $\Theta(t)$ is a step function. Find the average energy, $\langle\Psi(t)|H(t)|\Psi(t)\rangle$, as a function of time, assuming the system is in the ground state for $t < 0$.

(c) (5 pts) Find the probability the system is in the original ground state as a function of time.

1 (a) $|\alpha\rangle = \exp\{\underbrace{\alpha a^+ + \alpha^* a}_{\text{Hermitian}}\} |0\rangle$

$$= e^{\frac{1}{2}\alpha^*\alpha[a,a^+]} e^{\alpha a^+} \underbrace{e^{\alpha^* a}}_{\hookrightarrow 1} |0\rangle$$

$$= e^{\alpha^*\alpha} \underbrace{e^{-\frac{1}{2}\alpha^*\alpha} e^{\alpha a^+} |0\rangle}_{\text{normalized state}}$$

$$\langle\alpha|\alpha\rangle = e^{2\alpha^*\alpha}$$

$$|\beta\rangle = \exp\{\underbrace{\beta a^+ - \beta^* a}_{\text{anti-Hermitian}}\} |0\rangle$$

$$\langle\beta|\beta\rangle = \langle 0| e^{-\beta a^+ + \beta^* a} e^{\beta a^+ - \beta^* a} |0\rangle$$

$$= 1$$

(B) THE HARD WAY

$$H = E_0 a^+ a + J_0 \Theta(t)(a + a^+)$$

$$|\eta(t)\rangle_I = e^{i\phi(t)} \exp\left\{ \underbrace{\frac{-iJ_0}{\hbar} \int_0^t dt' \, e^{-iE_0 t'/\hbar}}_{\equiv -i\eta(t)} (a+a^+) \right\} |0\rangle$$

$$\langle \eta(t) | H(t) | \eta(t) \rangle = \langle \eta_I(t) | H_I(t) | \eta_I(t) \rangle$$

$$= \langle 0 | e^{i\eta(t)(a+a^+)} \left\{ E_0 a^+ a + J_0 a e^{-iE_0 t/\hbar} + J_0 a^+ e^{iE_0 t/\hbar} \right\} e^{-i\eta(t)(a+a^+)} |0\rangle$$

$$= E_0 \eta^* \eta + J_0 \left(-i\eta(t) e^{-iE_0 t/\hbar} + i\eta^*(t) e^{iE_0 t/\hbar} \right) |0\rangle$$

$$= E_0 \eta^*(t)\eta(t) + J_0 \left\{ i\eta^* e^{iE_0 t/\hbar} - i\eta(t) e^{-iE_0 t/\hbar} \right\}$$

$$\eta(t) = \frac{J_0}{\hbar} \int_0^t dt' \, e^{+iE_0 t'/\hbar}$$

$$= \frac{J_0}{\hbar} \left[e^{+iE_0 t/\hbar} - 1 \right] \frac{-i\hbar}{E_0}$$

$$= -\frac{iJ_0}{E_0} e^{+iE_0 t/2\hbar} \cdot \left(e^{iE_0 t/2\hbar} - e^{-iE_0 t/2\hbar} \right)$$

$$= \frac{2J_0}{E_0} \sin\left(E_0 t/2\hbar\right) e^{+iE_0 t/2\hbar}$$

$$\langle \eta | H | \eta \rangle = \left(4J_0^2/E_0\right) \sin^2\left(E_0 t/2\hbar\right) + \frac{2J_0^2}{E_0}\left(\sin E_0 t/2\hbar\right)\left\{ -i e^{-iE_0 t/2\hbar} + i e^{iE_0 t/2\hbar} \right\}$$

$$= 0$$

1b THE EASY WAY

$$H = E_0 a^+ a + J_0 (a + a^+)$$

$$\langle \psi(t>0) | H | \psi(t>0) \rangle = \text{constant}$$

$$= \langle \psi(t=0^-) | H(t=0^+) | \psi(t=0^-) \rangle$$

$$= \langle 0 | E_0 a^+ a + J_0 (a + a^+) | 0 \rangle$$

$$= 0$$

1c $|\eta(t)\rangle = e^{i\varphi(t)} \exp\{-i\eta(t)(a+a^+)\}|0\rangle$

$= e^{i\varphi t} e^{-\eta^*\eta/2} e^{-i\eta(t)a^+}|0\rangle$

$|\langle 0|\eta\rangle|^2 = e^{-\eta^*\eta}$

$|\eta| = \frac{2J_0}{E_0} \sin\frac{E_0 t}{2\hbar}$

From lecture notes:

$\eta(t) = \frac{-i}{\hbar}\int_{-\infty}^{t} dt'\, j(t')\, e^{iE_0t'/\hbar}$

$= \frac{J}{\mathcal{E}}\left(1 - e^{iE_0t/\hbar}\right)$

$|\eta| = \frac{2J}{E_0}\sin\left(E_0t/2\hbar\right)$

2. Consider the following matrix element,

$$\langle \boldsymbol{\alpha}, \boldsymbol{J_F}, \boldsymbol{M_F}| \left(\sum_{ijk} \boldsymbol{\epsilon_{ijk} A_i B_j C_k} \right) |\boldsymbol{\beta}, \boldsymbol{J_I}, \boldsymbol{M_I}\rangle,$$

where $\vec{\boldsymbol{A}}, \vec{\boldsymbol{B}}$ and $\vec{\boldsymbol{C}}$ are odd-parity vector operators (like momentum or position). The labels $\boldsymbol{J_I}, \boldsymbol{J_F}$ and $\boldsymbol{M_I}, \boldsymbol{M_F}$ refer to the total angular momenta and their $\boldsymbol{z}-$projections for the initial and final states. The labels $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ denote any other quantum numbers characterizing the initial and final states.

(a) (10 pts) If $\boldsymbol{J_I}$=1 and $\boldsymbol{M_I} = \boldsymbol{1}$, for what values of $\boldsymbol{J_F}$ and $\boldsymbol{M_F}$ might the matrix element be non-zero?

(b) (5 pts) If there is no spin, i.e. $\vec{\boldsymbol{J}}$ is also the total orbital angular momentum ($\vec{\boldsymbol{L}} = \vec{\boldsymbol{J}}$), and $\boldsymbol{J_I} = \boldsymbol{M_I} = \boldsymbol{1}$, for what values of $\boldsymbol{J_F}$ and $\boldsymbol{M_F}$ might the matrix element be non-zero?

(c) (10 pts) Now, consider the matrix element

$$\boldsymbol{\mathcal{A}_{M_I,M_F}} = \langle \boldsymbol{\alpha}, \boldsymbol{J_F}, \boldsymbol{M_F}|\boldsymbol{P_x^{(a)} P_y^{(b)}}|\boldsymbol{\beta}, \boldsymbol{J_I}, \boldsymbol{M_I}\rangle,$$

where $\boldsymbol{J_I} = \boldsymbol{3}$ and $\boldsymbol{J_F} = \boldsymbol{1}$. List all the values of $\boldsymbol{M_I}$ and $\boldsymbol{M_F}$ for which $\boldsymbol{\mathcal{A}_{M_I,M_F}}$ might be non-zero. Here, $\boldsymbol{P^{(a)}}$ is the momentum operator acting on some component $\boldsymbol{(a)}$ and $\boldsymbol{P^{(b)}}$ is the momentum operator acting on some component $\boldsymbol{(b)}$. E.g. $\boldsymbol{P^{(a)}}$ might be the momentum of the electrons and $\boldsymbol{P^{(b)}}$ might be the momentum of the protons.

2a) Operator is scalar

$J_f = J_i$, $M_i = M_f$

$J_f = 1$, $M_f = 1$

b) Operator is odd parity
Must give zero for initial/final states of same parity

NO STATES

c) $xy = R^2 \sin^2\theta \sin\varphi \cos\varphi$

$= \frac{R^2 \sin^2\theta}{4} i \left(e^{i\varphi} + e^{-i\varphi}\right)\left(e^{i\varphi} - e^{-i\varphi}\right)$

$= \frac{i R^2}{4} \sin^2\theta \left(e^{2i\varphi} - e^{-2i\varphi}\right)$

$\sim \quad T^2_2 - T^2_{-2}$

M_i	M_f
3	1
2	0
1	−1
0	X
−1	1
−2	0
−3	−1

3. Consider a beam of particles of momentum $\hbar k$ elastically scattering off three identical targets placed at the following positions:

$$\vec{R}_1 = (x = 0, y = 0, z = 0),$$
$$\vec{R}_2 = (x = R, y = 0, z = 0),$$
$$\vec{R}_3 = (x = -R, y = 0, z = 0).$$

The direction of the scattered particles is denoted in spherical coordinates, with θ describing the direction relative to the beam (z) axis, and ϕ measuring the direction relative to the x axis in the $x - y$ plane, i.e. if the wave number for the scattered particle is $\vec{k}^{(\mathrm{f})}$,

$$k_z^{(\mathrm{f})} = k^{(\mathrm{f})} \cos\theta, \quad k_x^{(\mathrm{f})} = k^{(\mathrm{f})} \sin\theta \cos\phi, \quad k_y^{(\mathrm{f})} = k^{(\mathrm{f})} \sin\theta \sin\phi.$$

(a) (10 pts) Consider scattering observed in the $x - z$ plane ($\phi = 0$). At what polar angles θ will the differential cross section disappear?

(b) (5 pts) Repeat for scattering observed in the $y - z$ plane ($\phi = 90°$).

3a) $1 + e^{-i\vec{q}\cdot R\hat{x}} + e^{i\vec{q}\cdot R\hat{x}} = 0$

$$1 + 2\cos(\underbrace{kR\sin\theta}) = 0$$

$$kR\sin\theta = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3} + 2n\pi \quad \text{or} \quad \frac{4\pi}{3} + 2n\pi$$

$n = 0, 1, \ldots$

zeros for

$$\theta = \sin^{-1}\left\{\frac{\frac{2\pi}{3} + 2n\pi}{kR}\right\} \text{ or } \sin^{-1}\left\{\frac{\frac{4\pi}{3} + 2n\pi}{kR}\right\}$$

$n = 0, 1, 2 \ldots$

until $\theta > \pi$

3b) $1 + 2\cos(0) = 0$

Never, no zeros

4. A particle of mass m experiences an attractive spherically symmetric potential,

$$V(r) = -\beta\delta(r-a),$$

where $\beta > 0$.

(a) (5 pts) In terms of a, and the electron mass m, what is the minimum value of β that results in a bound state?

(b) (10 pts) What is the cross section in the limit that the incident beam energy is zero.

(c) (5 pts) If a scattered wave in a large volume behaves as

$$\psi(\vec{q}, \vec{r}, t) \sim e^{i\vec{q}\cdot\vec{r} - i\omega t}, \ t \to \infty$$

in the outgoing limit (large time after interacting with potential), what is the relative probability,

$$\alpha = \frac{\rho(\vec{r}=0)}{\rho_0(\vec{r}=0)},$$

that it will ~~appear~~ be at the origin while interacting with the potential? Here ρ_0 is the probability density (per unit volume) in the absence of the potential, and ρ is the probability density with the potential in place. FYI: The ratio α would be the same if the boundary conditions specified an incoming plane wave, instead of matching to an outgoing plane wave.

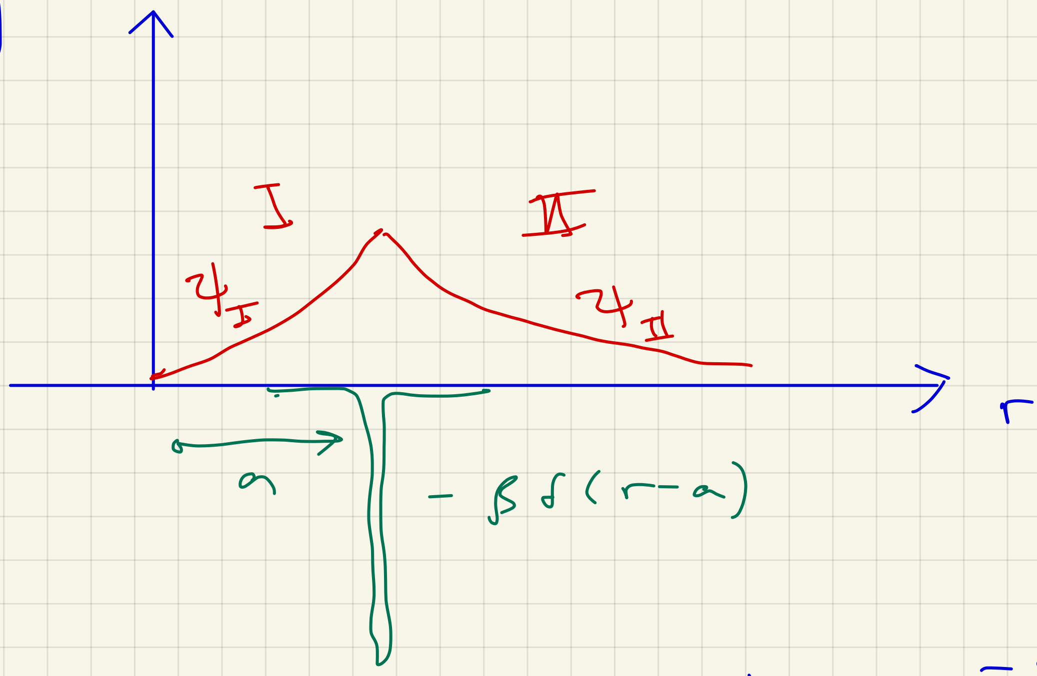
(d) (5 pts) Assume β is sufficiently large to bind a particle, and that the ground state energy is $-B$. For the ground state what is then probability density of finding the particle at $\vec{r} = 0$? Refer to this as $\rho_b(\vec{r} = 0)$? Given answer in terms of a and the binding energy B (or equivalently the decay wave number, $q \equiv \sqrt{2mB/\hbar^2}$). HINT: You don't need to solve for the binding energy!

(e) (10 pts) A small external potential is added,

$$V'(\vec{r}) = g\delta^3(\vec{r})\cos\omega t,$$

where $\hbar\omega > B$. What is the ionization rate? Express your answer in terms of m, B, g, α and $\rho_b(r=0)$.

4 a)



$$\psi_I = A \sinh qr \quad , \quad \psi_{II} = e^{-q(r-a)}$$

first B C: $\qquad A = 1/\sinh qa$

2nd B.C.: $\quad qA\cosh qa + q = \dfrac{2m\beta}{\hbar^2}$

$$q\frac{\cosh qa}{\sinh qa} + q = \frac{2m\beta}{\hbar^2}$$

Solution for $q \to 0$

$$\frac{2m\beta a}{\hbar^2} = 1 \quad , \quad \beta_{min} = \frac{\hbar^2}{2ma}$$

4b

$$-\frac{\hbar^2}{2m}\partial_r^2 \psi = (E - \beta\,\delta(r-a))\psi$$

$$\partial_r \psi_+ - \partial_r \psi_- = \frac{2m}{\hbar^2}\beta\psi$$

$$\psi_I = A\sin kr \quad , \quad \psi_{II} = \sin(kr+\delta)$$

$$\psi_I' = kA\cos kr \quad , \quad \psi_{II}' = k\cos(kr+\delta)$$

$$A\sin ka = \sin(ka+\delta)$$

$$\tilde{\beta} = \frac{2m\beta}{\hbar^2} \qquad k\cos(ka+\delta) - Ak\cos ka = -\tilde{\beta}A\sin ka$$

$$-\tilde{\beta}A\sin ka + Ak\cos ka = k\cos ka+\delta$$

$$\frac{\sin ka}{k\cos ka + \tilde{\beta}\sin ka} = \frac{1}{k}\tan(ka+\delta)$$

$$\delta = -ka + \tan^{-1}\left\{\frac{\sin ka}{\cos ka - \frac{\tilde{\beta}}{k}\sin ka}\right\}$$

$$\sigma = \frac{4\pi}{k^2}\sin^2\delta$$

$$\delta(k\to 0) = -ka + \tan^{-1}\left(\frac{ka}{1-\tilde{\beta}a}\right)$$

$$\sigma(k\to 0) = 4\pi a^2\left(-1 + \frac{1}{1-\tilde{\beta}a}\right)^2$$

$$= 4\pi\frac{\tilde{\beta}^2 a^4}{(1-\tilde{\beta}a)^2}$$

4c. At origin only s-wave contributes

$$R_\ell(r) = e^{i\delta}\,\frac{u_\ell(r)}{kr}$$

$$u_{\ell=0} = \begin{cases} A\sin kr, & r<a \\ \sin kr+\delta, & r>a \end{cases}$$

$$\delta = -ka + \tan^{-1}\left\{\frac{\sin ka}{\cos ka - \frac{\tilde{\beta}}{k}\sin ka}\right\}$$

$$A = \sin\left[\tan^{-1}\{\quad\}\right]\Big/\sin ka$$

$$= \frac{1}{\sin ka}\sqrt{1-\frac{1}{1+\{\quad\}^2}}$$

$$= \frac{1}{\sin ka}\,\frac{\{\quad\}}{\sqrt{1+\{\quad\}^2}}$$

$$= \frac{1}{\cos ka - \frac{\tilde{\beta}}{k}\sin ka}\,\frac{1}{\sqrt{1+\frac{\sin^2 ka}{(\cos ka - \tilde{\beta}/k\,\sin ka)^2}}}$$

$$A = \frac{1}{\sqrt{1-\frac{\tilde{\beta}}{k}\sin 2ka + \frac{\tilde{\beta}^2}{k^2}\sin^2 ka}}$$

$$|\psi(\vec{r}=0)|^2 = |A|^2 = \frac{1}{1-\frac{\tilde{\beta}}{k}\sin 2ka + \frac{\tilde{\beta}^2}{k^2}\sin^2 ka}$$

4 d) $\rho_b(r) = |\varphi(r)|^2$

$$|\varphi(r)|^2 = \frac{1}{Z r^2} \begin{cases} \frac{\sinh^2 qr}{\sinh^2 qa}, & r < a \\ e^{-2q(r-a)}, & r > a \end{cases}$$

$\nearrow$ normalization

$$\int 4\pi r^2 dr\, |\varphi(r)|^2 = 1$$

$$Z = 4\pi \left[\int_0^a dr \frac{\sinh^2 qr}{\sinh^2 qa} + \frac{1}{2q} \right]$$

$$|\varphi(r=0)|^2 = \frac{1}{Z} \frac{q^2}{\sinh^2 qa}$$

$$Z = \frac{2\pi}{q} + \frac{4\pi}{\sinh^2 qa} \int_0^a dr \left(\frac{\cosh 2qr - 1}{2} \right)$$

$$= \frac{2\pi}{q} + \frac{\pi}{\sinh^2 qa} \left(\frac{1}{q} \sinh 2qa - 2a \right)$$

$$|\varphi(r=0)|^2 = \frac{q^3}{2\pi \sinh^2 qa + \pi \sinh 2qa - 2\pi qa}$$

$$= \rho_b(r=0)$$

4e

$$\Gamma = \frac{\pi}{2\hbar} \left| \langle f | V | i \rangle \right|^2 \delta(E - (B + \hbar\omega))$$

$$\left| \langle f | V | i \rangle \right|^2 = \left| g \int d^3r\, \delta^3(r)\, \psi^*_{\vec{k}}(r)\, \psi_B(\vec{r}) \right|^2$$

$$= g^2 \rho_k(r=0)\, \rho_b(r=0)$$

$$= \frac{g^2 \alpha}{V} \rho_b(r=0)$$

$$\Gamma = \frac{\pi}{2\hbar} \frac{g^2}{V} \rho_b\, V \int \frac{d^3k}{(2\pi)^3} \alpha(k)\, \delta(E - (B + \hbar\omega))$$

$$\frac{dE}{dk} = \frac{\hbar^2 k}{m}$$

$$\Gamma = \frac{\pi}{2\hbar} g^2 \rho_b(r=0) \frac{1}{2\pi^2} \alpha(k) \frac{m}{\hbar^2 k} k^2$$

where $\frac{\hbar^2 k^2}{2m} = \hbar\omega - B$

or $k = \sqrt{\frac{2m}{\hbar^2}(\hbar\omega - B)}$

$$\Gamma = \frac{g^2 \alpha(k)\, mk}{4\pi\hbar^3} \rho_b(r=0)$$